Research Statement

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There is a classical connection between continued fraction (or "CF") expansions

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

and geodesic flow on the modular surface $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathcal{H}^2$. This overlap of symbolic dynamical systems, geometry, and number theory has inspired much of my research. My current research program essentially consists of three settings, summarized here:

- **Natural extensions and coding.**

  I defined the "finite building property" [2] for complex continued fractions, proved that several historical complex CF algorithms have this property, and showed that the domain of the natural extension of their (complex) Gauss maps can be described as a finite union of Cartesian products $K_i \times L_i \subset \mathbb{C} \times \mathbb{C}$.

  Jointly with Svetlana Katok [5], I investigated geodesic flow on quotients $\Gamma\backslash\mathbb{D}$ of the hyperbolic disk $\mathbb{D}$ by cocompact torsion free Fuchsian groups Γ using the "boundary maps" $f_{\bar{A}} : \partial\mathbb{D} \to \partial\mathbb{D}$ (see (5)). We showed that for a large class of parameters $\bar{A}$, the domain of the natural extension of $f_{\bar{A}}$ corresponds to a dynamical cross-section for the geodesic flow on $\Gamma\backslash\mathbb{D}$.

  Individually [3], I extended previous results about $f_{\bar{A}}$ that held only for parameters with "short cycles" to the class of "extremal" parameters—which were later used in [7]—and showed that every extremal parameter admits a dual.

- **Entropies of piecewise monotone circle maps.**

  Jointly with Svetlana Katok and Ilie Ugarcovici [6, 7, 8], I determined that for $\Gamma\backslash\mathbb{D}$ of genus g the entropy of $f_{\bar{A}}$ with respect to its unique smooth invariant measure $\mu_{\bar{A}}$ varies within the Teichmüller space $\mathcal{T}(g)$, taking any positive value less than a maximum that is achieved on the surface that admits a regular $(8g-4)$-gon. By contrast, we also proved that $h_{\mathrm{top}}(f_{\bar{A}})$ is constant both within $\mathcal{T}(g)$ and across all parameters $\bar{A}$.

  The proof that $h_{\mathrm{top}}(f_{\bar{A}})$ does not depend on $\bar{A}$ uses conjugation to maps of constant slope. In [9] we applied this technique to a two-parameter family of real continued fraction transformations $f_{a,b} : \mathbb{R} \to \mathbb{R}$ (see (4)), proving that $h_{\mathrm{top}}(f_{a,b})$ is constant on a large subset of the parameter space for (a, b).

- **Preservation and destruction of normality.**

  Jointly with Tomasz Downarowicz, I have proven that sampling digits of a "CF-normal" continued fraction expansion along a deterministic sequence with density between 0 and 1 will always result in a number that is not CF-normal (this extended an existing result that held only for arithmetic sampling). We also proved that the same holds when CF expansions are replaced by any shift system that has completely positive entropy but is not a Bernoulli shift.

The summaries in this statement assume some familiarity with the definitions of measure-theoretic and topological entropy for piecewise continuous maps; geodesic flow on quotients of the hyperbolic plane; Tecimüller space; Bernoulli shifts and topological Markov chains; shift systems with completely positive entropy; generic points for measures; joinings and (Furstenberg) disjointness of measures; and upper and lower density of natural sequences.

Contents

Complex continued fractions

Let $\mathbb{C} \times \mathbb{R}_+$ be the half-space model of $\mathcal{H}^3_{\mathbb{R}}$. In my work so far building analogues of results in $\mathcal{H}^2_{\mathbb{R}}$ for $\mathcal{H}^3_{\mathbb{R}}$, some features transfer very nicely, some encounter difficulties that I have successfully overcome, and some results are as-yet unknown.

Background. Combining terminology from Dani–Nogueira [17] and notation from Katok–Ugarcovici [27, 28], a function $\lfloor\cdot\rceil : \mathbb{C}\backslash\{0\} \to \mathbb{Z}[i]$ such that $|z - \lfloor z\rceil\,| \leq 1$ is called a *choice function*. Each choice function has an associated *fundamental set*

$$K = \overline{\{\, z - \lfloor z\rceil : z \in \mathbb{C}\,\}}.$$

For example, the "Hurwitz algorithm" [22] has $\lfloor z\rceil$ the closest Gaussian integer to z, and then K is the unit square centered at the origin. For any CF algorithm (that is, any choice function $\lfloor\cdot\rceil$), the *Gauss map* $g : K \to K$ given by

$$g(0) = 0, \qquad g(z) = \frac{-1}{z} - \left\lfloor \frac{-1}{z} \right\rceil \text{ for } z \neq 0 \tag{1}$$

can be used to construct a CF expansion of $z \in \mathbb{C}$ by $a_0 = \lfloor z\rceil$ and $a_n = \lfloor -1/g^{n-1}(z - a_0)\rceil$ for $n > 0$. If this $f^n(z) = 0$ for some n, then $z = [a_0; a_1, \dots, a_n]$ and the sequence terminates. The natural extension of g to $\mathbb{C} \times \mathbb{C}$ can be given by

$$G(z, w) = \left(\frac{-1}{z} - a, \frac{-1}{w} - a \right), \qquad a = \left\lfloor \frac{-1}{z} \right\rceil. \tag{2}$$

Natural extensions of Gauss maps for several real and complex continued fraction algorithms have been used to derive absolutely continuous invariant measures for the Gauss maps themselves.

New results on complex CF. In the case of real (a, b)-continued fractions, the orbits of the two discontinuity points of "$f_{a,b}$" from [27, 28] collide after finitely many iterations, and this "cycle property" is heavily used in the analysis of real-valued Gauss maps and their natural extensions. In the complex setting, $K \subset \mathbb{C}$ replaces the interval $[a, b)$, but since ∂K is not a finite set of points, tracking its orbit is significantly more complicated. Thus, I developed the "finite building property" to serve as a replacement for the cycle property.

> **Definition.** A continued fraction algorithm with Gauss map $g : K \to K$ has the *finite building property* if there exists a finite partition $\mathcal{P} = \{K_1, \dots, K_N\}$ of K with $N > 1$ such that each $g(K_i)$ is equal, up to measure zero, to some union of elements of $\mathcal{P}$.

In [2], I showed that each of the following continued fraction algorithms do satisfy the finite building property:

- the nearest integer or (Adolf) Hurwitz algorithm from [22].
- the nearest even integer, in which $\operatorname{Re}\lfloor z\rceil + \operatorname{Im}\lfloor z\rceil$ is always even [23].
- the nearest odd integer, in which $\operatorname{Re}\lfloor z\rceil + \operatorname{Im}\lfloor z\rceil$ is always odd.
- the "diamond algorithm" from [1].
- the "disk algorithm" described by Tanaka [34].
- the "shifted Hurwitz" algorithm described by Dani–Nogueira [17, Ex. 2.3#2].

Analogous to the "finite rectangular structure" of $\Omega_{a,b} \subset \mathbb{R}\times\mathbb{R}$ from [27, 28] and $\Omega_{\bar{A}} \subset \mathbb{S}\times\mathbb{S}$ from [29, 5, 6], I show that, for four of the algorithms above, the map G from (2) is bijective a.e. on a set with $\Omega \subset \mathbb{C}\times\mathbb{C}$ with "finite product structure."

Theorem ([2, Theorem 3.9])**.** *Consider an algorithm that satisfies the finite building property with partition $\{K_1,\dots,K_N\}$, and let $L_1,\dots L_N \subset \mathbb{C}$ be arbitrary closed sets such that the boundaries of each $K_i\times L_i$ have zero 2-dim. Lebesgue measure. The map G is bijective a.e. on the set*

$$\Omega := \bigcup_{i=1}^{N} K_i \times L_i$$

if and only if the following system holds:

$$L_i = \bigcup_{(a,j)\in\mathcal{A}_i} \left\{ \frac{-1}{w} - a : w \in L_j \right\} \qquad \text{for } 1 \le i \le N, \tag{3}$$

where $\mathcal{A}_i = \{\, (a,j) \in \mathbb{Z}[i]\times\{1,\dots,N\} : K_i \subset g(K_j \cap \langle a\rangle)\,\}$.

The sets $K_1,\dots,K_N$ are determined by hand from the CF algorithm (see [2, Section 3] for an example), but the process of finding the corresponding L_i involves experimental assistance: a computer iterates random points in $\mathbb{C}\times\mathbb{C}$ under G for a given complex continued fraction algorithm and then generate scatter plots approximating

$$\operatorname{proj}_2\big(\Omega \cap (K_i\times\mathbb{C})\big) = \{\, w : \exists\, z \in K_i \text{ s.t. } (z,w)\in\Omega \,\}$$

or its image under S. Figure 1 shows an approximation (left) of SL_1 for the nearest even algorithm—the computer is given the function $\lfloor z\rceil$ and the sets $K_1,\dots,K_8$—along with the actual set SL_1 (right of Figure 1). Once these L_i are hypothesized from the numerics, they can be rigorously shown to satisfy the system (3). See the proof of [2, Theorem 4.4] for details of this process with the nearest even algorithm.

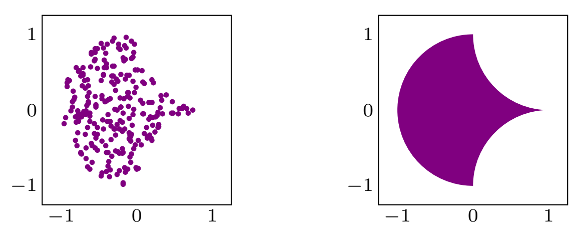

FIGURE 1. Determining $-1/L_1$ for the nearest even algorithm by experimentation.

SURFACES OF CONSTANT NEGATIVE CURVATURE

In [5], Svetlana Katok and I extended the results of Adler and Flatto [11] to a family of boundary maps $f_{\bar{A}}$ with parameters having short cycles, and in [3] I individually proved similar results for extremal parameters. In [6], Katok and Ugarcovici and I proved that the measure-theoretic entropy of $f_{\bar{A}}$ is flexible. In [7] we proved that topological entropy of $f_{\bar{A}}$ is rigid, and in [9] we applied the same techniques to part of the parameter space for the continued fraction maps $f_{a,b}$.

Background. The maps $T(z) = z+1$ and $S(z) = -1/z$ generate the group $\mathrm{PSL}(2,\mathbb{Z})$ and act on the half-plane $\mathcal{H}^2 = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$ with $\mathrm{d}s^2 = (\mathrm{d}x^2 + \mathrm{d}y^2)/y^2$. The standard fundamental domain for $\mathrm{PSL}(2,\mathbb{Z})$ is shown in Figure 2(a). For any parameters $a \le 0 \le b$ satisfying $b - a \ge 1$ and $-ab \le 1$, the real map

$$f_{a,b}(x) := \begin{cases} x+1 & \text{if } x < a \\ -1/x & \text{if } a \le x < b \\ x-1 & \text{if } x \ge b, \end{cases} \tag{4}$$

can be used to construct continued fraction expansions as described in [27, Section 2].

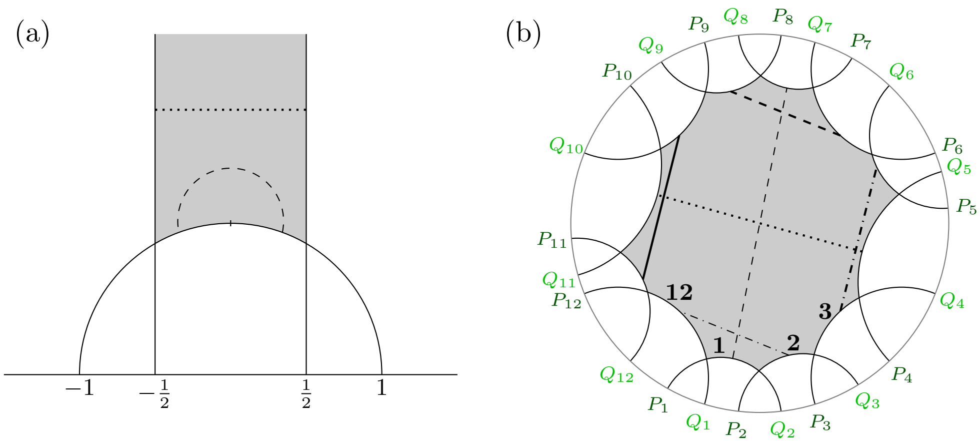

FIGURE 2. (a) Fundamental domain for the modular surface $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathcal{H}^2$; (b) fundamental $(8g-4)$-gon $\mathcal{F}$ for the surface $S = \Gamma\backslash\mathbb{D}$ with genus $g = 2$.

The "modular surface" $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathcal{H}^2$ is non-compact. An analogous setup that yields a compact, oriented surface S of genus $g \ge 2$ and constant negative curvature is the following: let Γ be a finitely-generated Fuchsian group of the first kind acting freely on the disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with $\mathrm{d}s^2 = (\mathrm{d}x^2 + \mathrm{d}y^2)/(1 - x^2 - y^2)^2$, and set $S = \Gamma\backslash\mathbb{D}$.

Adler and Flatto [11] describe an $(8g-4)$-sided polygon $\mathcal{F} \subset \mathbb{D}$ that serves as a fundamental polygon for $S = \Gamma\backslash\mathbb{D}$. An example is shown in Figure 2(b). The sides of $\mathcal{F}$ are labeled 1 through $8g-4$ counter-clockwise, and side i is glued to side $\sigma(i)$, where

$$\sigma(i) := \left\{ \begin{array}{ll} 4g - i \bmod (8g-4) & \text{if } i \text{ is odd} \\ 2 - i \bmod (8g-4) & \text{if } i \text{ is even.} \end{array} \right.$$

The polygon $\mathcal{F}$ is not necessarily regular, but it is related to a regular $(8g-4)$-gon centered at the origin (see [11, Figure 1]) by a homomorphism of $\bar{\mathbb{D}}$ whose existence is guaranteed by the Fenchel–Nielsen Theorem [35]. The Teichmüller space $\mathcal{T}(g)$ for $g \ge 2$ can be modeled as the space of marked canonical $(8g-4)$-gons in the unit disk $\mathbb{D}$, up to an isometry of $\mathbb{D}$.

Analogous to $z \mapsto z+1$ and $z \mapsto -1/z$ (which generate $\mathrm{PSL}(2,\mathbb{Z})$ and glue sides of the polygon in Figure 2(a) to each other), the generators of Γ are the maps $T_1, ..., T_{8g-4}$, where $T_i : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ is the Möbius transformation mapping side i to side $\sigma(i)$. The analogue of the continued fraction map uses a (multi-)parameter

$$\bar{A} = \{A_1, \dots, A_{8g-4}\}, \quad A_i \in [P_i, Q_i],$$

where P_i and Q_{i+1} in $\mathbb{S} := \partial\mathbb{D}$ are the endpoints of the infinite oriented geodesic containing side i, see Figure 2(b). For each $\bar{A}$, we define the *arithmetic map* $f_{\bar{A}} : \mathbb{S} \to \mathbb{S}$ by

$$f_{\bar{A}}(x) = T_i(x) \quad \text{if } x \in [A_i, A_{i+1}) \tag{5}$$

and the map $F_{\bar{A}}$ on $\mathbb{S}\times\mathbb{S}\setminus\Delta$, where $\Delta = \{\,(x,x) : x\in\mathbb{S}\,\}$, by

$$F_{\bar{A}}(u,w) = (T_i(u), T_i(w)) \quad \text{if } w\in[A_i, A_{i+1}). \tag{6}$$

The map $f_{\bar{A}}$ may be called the "generalized Bowen–Series boundary map" because, in our language, Bowen and Series [16] studied the case $\bar{A} = \bar{P}$. Later, Adler and Flatto [11] studied both $\bar{A} = \bar{P}$ and $\bar{A} = \bar{Q}$. If we identify the geodesic from u to w with a point in $\mathbb{S}\times\mathbb{S}$, the map $F_{\bar{A}}$ may also be considered as a map on geodesics.

There are two important classes of parameters: a parameter $\bar{A} = \{A_1,\dots,A_{8g-4}\}$ is said to satisfy the *short cycle property* if $f_{\bar{A}}(T_iA_i) = f_{\bar{A}}(T_{i-1}A_i)$ for all $1\le i\le 8g-4$, and a parameter $\bar{A}$ is called *extremal* if each $A_i\in\{P_i,Q_i\}$. These properties are preserved by the Fenchel–Nielsen homeomorphism $h\big|_{\mathbb{S}}$ (see [30]), so it makes sense to talk about a parameter $\bar{A}$ satisfying these properties in the Teichmüller space $\mathcal{T}(g)$.

Independent of the parameter $\bar{A}$, each polygon $\mathcal{F}$ has an associated *geometric map*

$$F_{\text{geo}}(u,w) = (T_i(u), T_i(w)) \quad \text{if the geodesic from } u \text{ to } w \text{ exits } \mathcal{F} \text{ through side } i, \tag{7}$$

whose domain is $\Omega_{\text{geo}} = \{\,(u,w)\in\mathbb{S}\times\mathbb{S} : \text{geodesic from } u \text{ to } w \text{ intersects } \mathcal{F}\,\}$.

New results on coding. The following theorem combines results from multiple papers. Adler and Flatto [11] proved parts (1) and (3) only for the cases $\bar{A} = \bar{P}$ and $\bar{A} = \bar{Q}$, and Katok and Ugarcovici [29, 30] proved parts (1) and (2) for parameters with short cycles. Jointly with Svetlana Katok [5], I proved part (3) for short cycles, and individually [3] I proved it for extremal parameters.

Theorem. *If $\bar{A}$ satisfies the short cycle property or is extremal, then*

(1) the map $F_{\bar{A}}$ is bijective a.e. on a set $\Omega_{\bar{A}}\subset\mathbb{S}\times\mathbb{S}$ that has a finite rectangular structure (the restriction $F_{\bar{A}}\big|_{\Omega_{\bar{A}}}$ is the natural extension *of the boundary map $f_{\bar{A}}$);*

(2) the set $\Omega_{\bar{A}}$ is the global attractor of $F_{\bar{A}}$, that is,

$$\Omega_{\bar{A}} = \bigcap_{n\ge 0} F_{\bar{A}}^n(\mathbb{S}\times\mathbb{S}\setminus\Delta);$$

(3) there exists a conjugacy between $F_{\text{geo}}:\Omega_{\text{geo}}\to\Omega_{\text{geo}}$ and the natural extension $F_{\bar{A}}\big|_{\Omega_{\bar{A}}}$. This implies that $\Omega_{\bar{A}}$ parameterizes a cross-section for geodesic flow, and the first return to the cross-section acts as $F_{\bar{A}}$.

The conjugacy mentioned in item (3) uses what Adler and Flatto call "bulges and corners," which are descriptive terms for components of $\Omega_{\text{geo}}\setminus\Omega_{\bar{A}}$ and $\Omega_{\bar{A}}\setminus\Omega_{\text{geo}}$.

Let $\gamma = uw$ be a geodesic on $\mathbb{D}$ with $(u,w)\in\Omega_{\bar{A}}$, and denote $(u_k,w_k) = F_{\bar{A}}^k(u,w)$ for all $k\in\mathbb{Z}$. The *arithmetic code* of γ is the sequence

$$[\gamma]_{\bar{A}} = (\dots, n_{-2}, n_{-1}, n_0, n_1, n_2, \dots)$$

where $n_k = \sigma(i)$ for the index i such that $w_k\in[A_i, A_{i+1})$. The first return to the cross-section of the flow along the projection of γ to $\Gamma\backslash\mathbb{D}$ corresponds to a left shift of the coding sequence $[\gamma]_{\bar{A}}$. In this way we associate to each $F_{\bar{A}}\big|_{\Omega_{\bar{A}}}$ a symbolic system—a shift on the closure of the set of all arithmetic codes.

The "future" of an arithmetic code $[uw]_{\bar{A}}$, that is, the terms n_k with $k\ge 0$, can be determined from w alone, but the "past" generally depends on both u and w. For continued fractions, the existence of "dual codes" (see [28, Sec. 5]) allows the digits in the past to be determined only from the endpoint u, and indeed a similar phenomenon can occur in the Fuchsian setting. Let $\phi(x,y) = (y,x)$. We say that two parameters $\bar{A}$ and $\bar{D}$ are *dual* if $\phi(\Omega_{\bar{A}}) = \Omega_{\bar{D}}$ and $\phi(F_{\bar{A}}^{-1}(p)) = F_{\bar{D}}(\phi(p))$ for all $p = (u,w)\in\Omega_{\bar{A}}$ with $u\notin\bar{D}$.

Theorem ([5, Theorem 9.2])**.** *If $\bar{A} = \{A_i\}$ and $\bar{D} = \{D_i\}$ are dual and $(u, w) \in \Omega_{\bar{A}}$, then the arithmetic code $[\gamma]_{\bar{A}}$ of the geodesic $\gamma = uw$ satisfies*

- *for $k \geq 0$, $n_k = \sigma(i)$ such that $f_{\bar{A}}^{k}(w) \in [A_i, A_{i+1})$, and*
- *for $k < 0$, $n_k = i$ such that $f_{\bar{D}}^{-k+1}(u) \in [D_i, D_{i+1})$.*

Unfortunately, dual codes do not exists for short cycles [5, Proposition 9.3]. This was initially a primary motivation for my study of the extremal cases, which lead to the following positive result:

Theorem ([3, Theorem 25])**.** *If $\bar{A}$ is extremal, then there exists a parameter $\bar{D}$ (not necessarily extremal) such that $\bar{A}$ and $\bar{D}$ are dual.*

Expressions for $D_1, ..., D_{8g-4}$ are described explicitly in [3, Proposition 5 and Equation 13].

New results on entropy. Results in the cocompact settings are presented first. For the non-cocompact setting (continued fractions), see page 7.

Recall the generalized Bowen–Series boundary maps $f_{\bar{A}}$ from (5). We study two dynamical invariants for these maps: the topological entropy and the measure-theoretic entropy with respect to a smooth (that is, Lebesgue-equivalent) measure. Our two main results are rigidity of the topological entropy and flexibility of the measure-theoretic entropy.

Theorem ([6, Theorems 1-2])**.** *Let $S = \Gamma\backslash\mathbb{D}$ be a surface of genus $g \geq 2$, and let $\bar{A}$ be extremal or satisfy the short cycle property.*

(1) ***Entropy formula:*** $h_{\mu_{\bar{A}}}(f_{\bar{A}}) = \pi \cdot \dfrac{\text{Area}(\mathcal{F})}{\text{Perimeter}(\mathcal{F})}$.

(2) ***Maximum:*** *Among all surfaces in the Teichmüller space $\mathcal{T}(g)$, the maximum value H of $h_{\mu_{\bar{A}}}(f_{\bar{A}})$ is achieved on the surface for which $\mathcal{F}$ is regular.*

(3) ***Flexibility:*** *For any $h \in (0, H]$ there exists $\mathcal{F} \in \mathcal{T}(g)$ such that $h_{\mu_{\bar{A}}}(f_{\bar{A}}) = h$.*

The proof of the entropy formula relies on the coding of geodesic flow via F_{geo} and the cross-section described above. Using this realization along with Abramov's Formula and the Ambrose–Kakutani Theorem, we have from [5, Proposition 10.1] (after correcting a constant factor) that

$$h_{\mu_{\bar{A}}}(f_{\bar{A}}) = h_{\nu_{\bar{P}}}(F_{\bar{P}}) = \frac{\pi^2(4g-4)}{\int_{\Omega_{\bar{P}}} \mathrm{d}\nu} = \pi \cdot \frac{\text{Area}(\mathcal{F})}{\int_{\Omega_{\bar{P}}} \mathrm{d}\nu},$$

where g is the genus of S and $\text{Area}(\mathcal{F}) = 2\pi(2g-2)$ comes from the Gauss–Bonnet formula. We then use the geometric map $F_{\text{geo}} : \Omega_{\text{geo}} \to \Omega_{\text{geo}}$ to show that $\int_{\Omega_{\bar{P}}} \mathrm{d}\nu$ is equal to the (hyperbolic) perimeter of $\mathcal{F}$, thus proving the entropy formula.

From the formula for $h_{\mu_{\bar{A}}}(f_{\bar{A}})$, the maximum result follows immediately from the fact that for a fixed area the perimeter of a polygon is minimized by the regular polygon. The flexibility result also follows, although a rigorous proof involves manipulation for the perimeter of $\mathcal{F}$ using Fenchel–Nielsen coordinates.

To analyze $h_{\text{top}}(f_{\bar{A}})$, we use very different machinery. If $\bar{A}$ is extremal, then $f_{\bar{A}}$ is Markov with partition $I_1, \dots, I_{16g-8}$ given by

$$I_{2i-1} = [P_i, Q_i], \qquad I_{2i} = [Q_i, P_{i+1}], \qquad i = 1, \dots, 8g-4,$$

([3, Proposition 19]; first shown for $\bar{A} = \bar{P}$ in [16, Lemma 2.5] and [11, Theorem 6.1]), which leads to a direct formula for $h_{\text{top}}(f_{\bar{A}})$ as the log of the maximal eigenvalue of the transition matrix $M_{\bar{A}}$ in these cases. As we move in the Teichmüller space $\mathcal{T}(g)$, the partition of $\mathbb{S}$ into $16g-8$ intervals above remains Markov with the same transition matrix, therefore $h_{\text{top}}(f_{\bar{A}})$ does not change as we move in $\mathcal{T}(g)$. Surprisingly, however, the map $f_{\bar{A}}$ also has this same topological entropy even for choices of $\bar{A}$ that fail to be Markov:

Theorem ([7, Theorem 1]). ***Rigidity:*** *For any polygon $\mathcal{F} \in T(g)$ and any parameter $\bar{A}$ with $A_i \in [P_i, Q_i]$, the map $f_{\bar{A}} : \mathbb{S} \to \mathbb{S}$ has topological entropy*

$$h_{\text{top}}(f_{\bar{A}}) = \log\big(4g - 3 + \sqrt{(4g-3)^2 - 1}\,\big).$$

The proof of rigidity involves an in-depth analysis of the maps $\psi_{\bar{P}} : \mathbb{S} \to \mathbb{S}$ and $\psi_{\bar{Q}} : \mathbb{S} \to \mathbb{S}$ that conjugate $f_{\bar{P}}$ and $f_{\bar{Q}}$, respectively, to maps with constant slope (the construction of these conjugacies is described by Parry [32]). We show that for any parameter $\bar{A}$, regardless of whether it is extremal or has short cycles, the conjugation $\psi_{\bar{P}} \circ f_{\bar{A}} \circ \psi_{\bar{P}}^{-1}$ (yes, with $\bar{P}$ and $\bar{A}$ as written) has constant slope $4g - 3 + \sqrt{(4g-3)^2 - 1}$.

As a corollary, the measure-theoretic entropy of $f_{\bar{A}}$ with respect to its smooth invariant measure $\mu_{\bar{A}}$ is always strictly less than the topological entropy of $f_{\bar{A}}$.

The details of the rigidity proof for $h_{\text{top}}(f_{\bar{A}})$ are quite technical [7, Sections 4-5 and Appendix A], but the general principles are also used for in [9] for the map $f_{a,b} : \mathbb{R} \to \mathbb{R}$ from (4), and this is summarized below.

Theorem ([9, Theorem 1]). *For any parameters $(a, b) \in \mathcal{S} := [-1, -\frac{1}{2}] \times [\frac{1}{2}, 1]$, the topological entropy of $f_{a,b} : \mathbb{R} \to \mathbb{R}$ is $\log(\frac{1+\sqrt{5}}{2})$.*

For convenience, we first use the homeomorphism $k(x) = x/(1+|x|)$ from $\bar{\mathbb{R}}$ to $[-1, 1]/\sim$ with ± 1 identified to conjugate $f_{a,b}$ to the map

$$\widetilde{f}_{a,b}(x) := k \circ f_{a,b} \circ k^{-1}(x) = \begin{cases} \widetilde{T}(x) & \text{if } -1 \le x < \frac{a}{1-a} \\ \widetilde{S}(x) & \text{if } \frac{a}{1-a} \le x < \frac{b}{1+b} \\ \widetilde{T}^{-1}(x) & \text{if } \frac{b}{1+b} \le x \le 1, \end{cases} \tag{8}$$

where $\widetilde{T} := k \circ T \circ k^{-1}$ and $\widetilde{S} := k \circ S \circ k^{-1}$ are conjugations of $T(x) = x+1$ and $S(x) = -1/x$. Each $\widetilde{f}_{a,b} : [-1, 1] \to [-1, 1]$ is a piecewise monotone map with two discontinuity points.

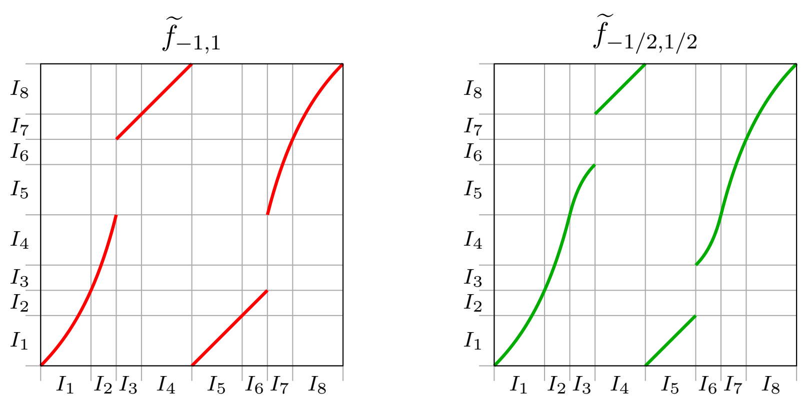

FIGURE 3. Plots of $\widetilde{f}_{-1,1}$ and $\widetilde{f}_{-1/2,1/2}$ with their (shared) Markov partition of $[-1, 1]$.

The maps $\widetilde{f}_{-1,1}$ and $\widetilde{f}_{-1/2,1/2}$ are each piecewise monotone, piecewise continuous, topologically transitive, and Markov with respect to *the same partition* $\{I_1, \ldots, I_8\}$ of $[-1, 1]$ (see Figure 3). We can directly compute the transition matrices $M_{-1,1}$ and $M_{-1/2,1/2}$, their eigenvalues, and thus the topological entropy of these two maps. Although $M_{-1,1}$ and $M_{-1/2,1/2}$ are not equal, they have the same maximal eigenvalue $\lambda = \frac{1+\sqrt{5}}{2}$ (and the same corresponding eigenvector). Therefore

$$h_{\text{top}}(\widetilde{f}_{-1,1}) = h_{\text{top}}(\widetilde{f}_{-1/2,1/2}) = \log(\tfrac{1+\sqrt{5}}{2}).$$

In the case where $\widetilde{f}_{a,b}$ has a Markov partition $\{I_1, ..., I_N\}$ (thus, in particular, for $\widetilde{f}_{-1,1}$ and $\widetilde{f}_{-1/2,1/2}$ with $N = 8$), the following construction due to Parry [32] gives an increasing homeomorphism $\psi_{a,b}$ such that $\psi_{a,b} \circ \widetilde{f}_{a,b} \circ \psi_{a,b}^{-1}$ has constant slope $e^{h_{\text{top}}(\widetilde{f}_{a,b})}$.

- First, define the probability measure $\rho_{a,b}$ on the shift space $X_{a,b} \subset \{1, ..., N\}^{\mathbb{N}}$ by
$$\rho_{a,b}\big(C_{a,b}(\omega_0, \dots, \omega_n)\big) = \lambda^{-n}\, v_{\omega_n},$$
where λ and $v = (v_1, ..., v_N)$ are the maximal eigenpair of the Markov transition matrix for $\widetilde{f}_{a,b}$, and $C_{a,b}(\omega)$ is a symbol cylinder.
- Define the Borel measure $\rho'_{a,b}$ on $[-1, 1]$ as the push-forward of $\rho_{a,b}$ via the essentially bijective "symbolic coding map" $(\omega_0, \omega_1, \omega_2, \dots) \mapsto \bigcap_{i=0}^{\infty} \widetilde{f}_{a,b}^{-i}(I_{\omega_i})$.
- The conjugacy is given by
$$\psi_{a,b}(x) := -1 + 2 \cdot \rho'_{a,b}\big([-1, x]\big).$$

This process gives us maps $\psi_{-1,1}$ and $\psi_{-1/2,1/2}$ that will conjugate $\widetilde{f}_{-1,1}$ and $\widetilde{f}_{-1/2,1/2}$, respectively, to maps of constant slope. A priori, the conjugacies are unrelated, but in fact

Theorem ([9, Theorem 6]). *For all $x \in [-1, 1]$, $\psi_{-1,1}(x) = \psi_{-1/2}(x)$.*

This is equivalent to the claim $\rho'_{-1,1}(I_{-1,1}(\omega)) = \rho'_{-1/2,1/2}(I_{-1,1}(\omega))$ for all $(-1, 1)$-admissible words ω, where
$$I_{a,b}(\omega_0) = I_{\omega_0}, \qquad I_{a,b}(\omega_0, \omega_1, \dots, \omega_n) = I_{\omega_0} \cap \widetilde{f}_{a,b}^{-1}(I_{a,b}(\omega_1, \dots, \omega_n))$$
is a "cylinder interval" (recall $I_i \subset [-1, 1]$ is an element of the Markov partition).

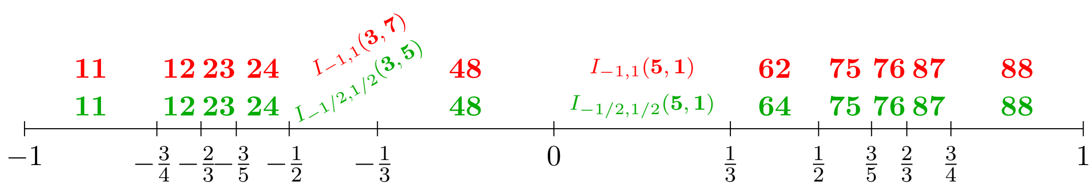

FIGURE 4. Rank-two cylinder intervals for $\widetilde{f}_{-1,1}$ (red) and $\widetilde{f}_{-1/2,1/2}$ (green) coincide.

Although $\rho'_{-1,1}(I_{-1,1}(\omega_0, ..., \omega_n)) = \lambda^{-n}\, v_{\omega_n}$ by construction, there is initially no way to calculate $\rho'_{-1/2,1/2}(I_{-1,1}(\omega))$. Fortunately, essentially every cylinder interval $I_{-1,1}(\omega)$ is exactly equal to a cylinder interval $I_{-1/2,1/2}(\tau)$ for some $(-\frac{1}{2}, \frac{1}{2})$-admissible word $\tau = (\tau_0, ..., \tau_n)$. Figure 4 shows this for $n = 1$. What we need is a "recoding process" to construct τ from ω.

The four words
$$3751, \quad 3762, \quad 6237, \quad 6248$$
are "bad" in the sense that they are $(-1, 1)$-admissible but not $(-\frac{1}{2}, \frac{1}{2})$-admissible. However,
$$I_{-1,1}(3, 7, 5, 1) = (\widetilde{f}_{-1,1}\big|_{I_3})(\widetilde{f}_{-1,1}\big|_{I_7})(\widetilde{f}_{-1,1}\big|_{I_5})(I_1) = \widetilde{S}\widetilde{T}\widetilde{S}(I_1) \quad \text{and}$$
$$I_{-1/2,1/2}(3, 5, 1, 1) = (\widetilde{f}_{-1/2,1/2}\big|_{I_3})(\widetilde{f}_{-1/2,1/2}\big|_{I_5})(\widetilde{f}_{-1/2,1/2}\big|_{I_1})(I_1) = \widetilde{T}^{-1}\widetilde{S}\widetilde{T}^{-1}(I_1),$$
and the equality $\widetilde{S}\widetilde{T}\widetilde{S} = \widetilde{T}^{-1}\widetilde{S}\widetilde{T}^{-1}$, proves that $I_{-1,1}(3, 7, 5, 1) = I_{-1/2,1/2}(3, 5, 1, 1)$, thus providing a recoding of 3751. For longer ω, we induct on the number of occurrences of the four "bad" words above.

Why these four substitutions (and partial versions such as $37 \to 35$) are enough to recode any ω to a corresponding τ is explained in [9, Section 2.2] along with the fact that $v_{\omega_n} = v_{\tau_n}$ after the recoding.

Once we have proven $\psi_{-1,1} = \psi_{-1/2,1/2}$, we can relabel this map as simply ψ. For any $(a,b) \in [-1,-\frac{1}{2}] \times [\frac{1}{2},1]$, the map $\psi \circ \widetilde{f}_{a,b} \circ \psi^{-1}$ will have constant slope $\frac{1+\sqrt{5}}{2}$ on the interval $\psi([-1,k(a)]) \subset \psi([-1,-\frac{1}{3}])$ because $\widetilde{f}_{a,b}$ acts as $\widetilde{T}$ on the interval $[-1,k(a)] \subset [-1,-\frac{1}{3}]$ and $\widetilde{f}_{-1/2,1/2}$ (which we know ψ conjugates to a map with slope $\frac{1+\sqrt{5}}{2}$) acts as $\widetilde{T}$ on all of $k([-\infty,-\frac{1}{2}]) = [-1,-\frac{1}{3}]$. Similarly, $\psi \circ \widetilde{f}_{a,b} \circ \psi^{-1}$ is linear with slope $\frac{1+\sqrt{5}}{2}$ on $\psi([k(a),k(b)])$ and on $\psi([k(b),1])$ because of where $\widetilde{f}_{a,b}$ acts by $\widetilde{S}$ and $\widetilde{T}^{-1}$.

Recalling that ψ is an increasing homeomorphism, we now have that $\psi \circ \widetilde{f}_{a,b} \circ \psi^{-1}$ is linear with slope $\frac{1+\sqrt{5}}{2}$ on all of

$$\psi([-1,k(a)]) \cup \psi([k(a),k(b)]) \cup \psi([k(b),1]) = \psi([-1,1]) = [-1,1].$$

Since $\widetilde{f}_{a,b}$ is conjugate to a map with constant slope $\frac{1+\sqrt{5}}{2}$, we have $h_{\text{top}}(\widetilde{f}_{a,b}) = \log(\frac{1+\sqrt{5}}{2})$.

Preservation and destruction of normality

Wall [36] showed in 1949 that arithmetic sequences "preserve normality," meaning that if $0.d_1d_2d_3\ldots$ is normal in some base then $0.d_\ell d_{\ell+m} d_{\ell+2m} d_{\ell+3m}\ldots$ also is normal. By contrast, normality is "destroyed" (see below) when one uses continued fraction expansions instead of decimal or base-b expansions.

Background. Let (X,μ,σ) be a one-sided shift on a finite or countable alphabet, with μ a fixed shift-invariant probability measure. A sequence $a \in X$ is called *μ-normal* if $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \phi(\sigma^i a) = \int \phi \, \mathrm{d}\mu$ for every continuous $\phi : X \to \mathbb{R}$.

> **Definition.** Let $S = (s_1, s_2, \ldots)$ be an increasing sequence of natural numbers. The sequence S *preserves μ-normality* if for any μ-normal sequence $a = (a_1, a_2, ...) \in X$ the restriction
>
> $$a|_S = (a_{s_1}, a_{s_2}, a_{s_3}, \ldots)$$
>
> is also μ-normal. The sequence S *destroys μ-normality* if for any μ-normal a the restriction $a|_S$ is not μ-normal.

Destruction of normality is much stronger than "not preserving normality," for which it would be sufficient to have just one counterexample a with $a|_S$ not normal.

Two particular types of μ-normality are worth mentioning separately:

- Classical normality in base b is the case where $X = \{0,...,b-1\}^{\mathbb{N}}$ and μ is the uniform Bernoulli measure on X. This corresponds to the Lebesgue measure on $[0,1]$ with $x \mapsto b\cdot x \bmod 1$ acting as a digit shift.
- Continued-fraction normality (CF-normality) is the case where μ is the measure on $\mathbb{N}^{\mathbb{N}}$ corresponding to the the Gauss measure $\frac{\mathrm{d}x}{(1+x)\ln 2}$ on $[0,1]$, which is invariant for the Gauss map $x \mapsto \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$ that shifts classical CF digits. We will call this measure on $\mathbb{N}^{\mathbb{N}}$ the *Gauss measure* as well.

In the 1970s, Kamae and Weiss [37, 24] generalized the 1949 result of Wall to the following: a sequence S with positive lower density preserves classical normality if and only if it is deterministic (see below). In 2016, Heersink and Vandehey [21] showed that non-trivial arithmetic progressions destroy CF-normality.

New results on normality. Jointly with Tomasz Downarowicz [4], I have extended the Heersink–Vandehey result to hold for any sequence S that is *deterministic*—meaning that all measures quasi-generated by S have entropy zero—and *essential*—meaning its lower density is less than 1 and its upper density is strictly positive.

Theorem ([4, Theorem 7])**.** *Essential deterministic sequences destroy CF-normality. That is, if $x = 1/(a_1 + 1/(a_2 + \cdots))$ is CF-normal and $S = (s_1, s_2, \dots)$ is an essential deterministic sequence then $1/(a_{s_1} + 1/(a_{s_2} + \cdots))$ is **not** CF-normal.*

Instead of Vandehey's "augmented Gauss map," our proof uses a joining of the Gauss measure μ on $\mathbb{N}^{\mathbb{N}}$ and a measure ν on $\{0,1\}^{\mathbb{N}}$ quasi-generated by S. Both proofs rely on the fact that

$$\mu([1, *^n, 1]) < \mu([1, 1]) \quad \text{for all } n \geq 1,$$

where $[1, *^n, 1]$ with $n \geq 0$ denotes the set of all numbers whose continued fraction expansion starts with 1 and has 1 as the $(n+2)^{\text{nd}}$ digit.

Let $x = 1/(a_1 + 1/(a_2 + \cdots))$ be CF-normal and denote by a the sequence $(a_1, a_2, \dots)$ of the coefficients in the continued fraction expansion of x. Consider the "double sequence"

$$\begin{pmatrix} a \\ \mathbb{1}_S \end{pmatrix} = \begin{pmatrix} a_1, & a_2, & a_3, & \dots \\ \mathbb{1}_S(1), & \mathbb{1}_S(2), & \mathbb{1}_S(3), & \dots \end{pmatrix},$$

which we view as one sequence whose entries are two-element columns. The block $(1,1)$ occurs in $x|_S$ if and only if a double block of the form

$$B_n = \begin{pmatrix} 1, & *^n, & 1 \\ 1, & 0^n, & 1 \end{pmatrix}$$

occurs in $\left(\begin{smallmatrix} a \\ \mathbb{1}_S \end{smallmatrix}\right)$ for some $n \geq 0$.

By analyzing the asymptotic growth of the number of occurrences of $\left(\begin{smallmatrix} 1, & *^n, & 1 \\ 1, & 0^n, & 1 \end{smallmatrix}\right)$, we show that the limit frequency

$$\mathsf{fr}_{a|_S}(1,1) = \lim_{N\to\infty} \sum_{n=0}^{\infty} \frac{\#\{1 \leq s_j \leq N : a_{s_j} = a_{s_{j+1}} = 1, s_{j+1} - s_j = n+1\}}{\#(\{1, \dots, N\} \cap S)}$$

does not equal $\mu([1,1])$, which implies that the restriction $a|_S$ is not CF-normal.

Moving beyond continued fractions, we extended the previous result to the following:

Theorem ([4, Theorem 4])**.** *If (X, μ, σ) is a shift system over a finite or countable alphabet that has completely positive entropy but is* not *a Bernoulli shift, then any essential deterministic sequence destroys μ-normality.*

In this setting we must replace the 2-digit block $(1,1)$ used in the CF-normality proof by some block-with-positions for which "spreading apart" the positions always causes the measure of the corresponding cylinder to decrease.

Indeed, the key property of Bernoulli shifts which helps in μ-normality preservation is that the Bernoulli measure of a cylinder does not decrease when the positions of the fixed symbols are spread apart (the Bernoulli measure depends only on the symbols and not on the positions at all). Any non-Bernoulli shift with completely positive entropy will have a block-with-positions with the above property, and this allows us to use the key ideas from the CF-normality proof in this more general case.

Current and future work

Many open questions and related topics have arisen during the course of the work presented here. I have four main areas of potential progress, and I am actively working on items 1 and 4 below.

1. Entropies of continued fraction maps. As discussed already here, the topological entropy of the continued fraction map $f_{a,b}$ is always $\log(\frac{1+\sqrt{5}}{2})$ within the square $[-1,-\frac{1}{2}]\times[\frac{1}{2},1]$. However, for other points in the full parameter space

$$\mathcal{P} = \{\,(a,b)\in\mathbb{R}^2 : a\le 0\le b,\ b-a\ge 1,\ -ab\le 1\,\},$$

direct computations demonstrate other values. For example, $h_{\mathrm{top}}(f_{-1,0}) = \log(\kappa)\approx 0.382$, where $\kappa^3-\kappa^2-1=0$. Experimental evidence suggests that $h_{\mathrm{top}}(f_{a,b})$ is flexible on the full parameter space:

Conjecture. *Let $\varphi = \frac{1+\sqrt{5}}{2}$, and let $\kappa > 0$ satisfy $\kappa^3-\kappa^2-1=0$.*
(1) The minimum and maximum of $h_{\mathrm{top}}(f_{a,b})$ are $\log(\kappa)$ and $\log(\varphi)$, respectively.
(2) For any $h\in[\log(\kappa),\log(\varphi)]$ there exists $(a,b)\in\mathcal{P}$ for which $h_{\mathrm{top}}(f_{a,b}) = h$.

The measure-theoretic entropy of the Gauss map $\widehat{f}_{a,b}$ (the first return of $f_{a,b}$ to the interval $[a,b]$) is known [28, Theorem 6.2] to be

$$h_{\widehat{\mu}_{a,b}}(\widehat{f}_{a,b}) = \frac{\pi^2}{3\,K_{a,b}}, \qquad K_{a,b} := \iint_{\widehat{D}_{a,b}} \frac{\mathrm{d}x\mathrm{d}y}{(1+xy)^2},$$

where $\widehat{D}_{a,b}\subset[-1,1]\times[-1,1]$ is the domain of the natural extension of $\widehat{f}_{a,b}$. An explicit formula for $K_{a,b}$ is given in [28, Theorem 7.1] for the case $1\le\frac{-1}{a}\le b+1$, but in the full parameter space the value of $K_{a,b}$ is not always known. A direct formula for $K_{a,b}$ could be very helpful in proving the following:

Conjecture. *Let $\varphi = \frac{1+\sqrt{5}}{2}$.*
(1) The maximum of $h_{\widehat{\mu}_{a,b}}(\widehat{f}_{a,b})$ is $\frac{\pi^2}{3\log(1+\varphi)}$.
(2) For any $h\in(0,\frac{\pi^2}{3\log(1+\varphi)}]$ there exists $(a,b)\in\mathcal{P}$ for which $h_{\widehat{\mu}_{a,b}}(\widehat{f}_{a,b}) = h$.

2. Generic parameters in the cocompact setting. With the notable exception of [7], most results on boundary maps $f_{\bar{A}}:\mathbb{S}\to\mathbb{S}$ require the parameters $\bar{A}$ to have the short cycle property or be extremal, and the proofs for each of these cases do not readily apply to other cases. For example, the realization of the natural extension of $f_{\bar{A}}$ as a two-dimensional map $F_{\bar{A}}|_{\Omega_{\bar{A}}}$ requires a precise description of the domain $\Omega_{\bar{A}}\subset\mathbb{S}\times\mathbb{S}$ with finite rectangular structure, and this is not currently known for generic parameters $\bar{A}$.

Two major research goals in this Fuchsian setting are to confirm that the results for these special cases of parameters also apply to generic parameters $\bar{A}$ and to simplify or unify the proofs for these cases (rather than relying on separate proofs of very similar results).

3. Preservation of normality. Within the class of shifts that have completely positive entropy, Bernoulli shifts are exactly those that preserve μ-normality, while all other shifts destroy μ-normality. However, outside this class we cannot make such implications. In fact, there are explicit counterexamples showing that μ-normality preservation is independent of whether μ is disjoint from the measures quasi-generated by S. So in general we can say only that disjointness implies simple μ-normality preservation (part (1) of the conjecture above).

Two ambitious tasks would be to state precisely what conditions on a sequence S are necessary for (full or simple) μ-normality preservation given a measure μ and to state a "checkable" sufficient condition for a pair (μ,S) that guarantees that S preserves μ-normality. This second task is already done for systems with completely positive entropy but is still open for many other classes of systems, including those with ergodic measures of entropy zero.

4. Circle-normality with Følner sequences. The definition of μ-normality presented earlier applies to a shift system (X, μ, σ). Since $\sigma^n(\sigma^m(x)) = \sigma^{n+m}(x)$, this is an action by the semigroup $(\mathbb{N}, +)$.

The concept of normality can be extended to actions by other (semi)groups G on a topological space (X, μ) as follows: given a Følner sequence $(F_n)_{n\geq 1}$, a point $x \in X$ is (F_n)-*generic for* μ if

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} \phi(gx) = \int \phi \, \mathrm{d}\mu$$

for every continuous $\phi : (X, \mu) \to \mathbb{R}$. A *Følner sequence* is an infinite sequence of finite subsets $F_1, F_2, ...$ of G that satisfies $\lim_{n\to\infty} |gF_n \cap F_n| \,/\, |F_n| = 1$ for all $g \in G$.

In particular, consider the $(\mathbb{N}, \times)$-action $x \mapsto nx \bmod 1$, and say that a point $x \in [0, 1]$ is *circle-*(F_n)*-normal* if it is (F_n)-generic for the Lebesgue measure.

> **Conjecture.** *Let (F_n) be a Følner sequence in $(\mathbb{N}, \times)$ such that $F_n \subset F_{n+1}$ and $\bigcup_{n\geq 1} F_n = \mathbb{N}$ and each F_n is the set of divisors of some $L_n \in \mathbb{N}$. Then Lebesgue-a.e. point $x \in [0, 1]$ is circle-(F_n)-normal.*

For this and several related statements and questions, it is helpful to model $(\mathbb{N}, \times)$ as an infinite-dimensional lattice $\bigoplus_{p\in\mathbb{P}} (\mathbb{N} \cup \{0\}, +)$ via the isomorphism

$$2^{k_1} 3^{k_2} 5^{k_3} 7^{k_4} 11^{k_5} \cdots \quad \mapsto \quad (k_1, k_2, k_3, \ldots),$$

see [13, Section 6]. In this model, the desired Følner sets are right rectangular prisms with one corner at the origin, allowing arithmetic problems to become (high-dimensional) geometric problems.

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