# Math 1433

11 December 2023

**Warm-up:**

For which values of $p$ does

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - p\, I_{2\times 2}$$

have an inverse?

Any system of linear equations can be represented as a matrix equation

$$A\vec{x} = \vec{b}.$$

If $A^{-1}$ exists, then

$$\vec{x} = A^{-1}\vec{b}$$

is the unique solution to the system.

But what if $A^{-1}$ doesn't exist?

- This could be because number of variables ≠ number of equations.
- This could be because $\det(A) = 0$.

Last time

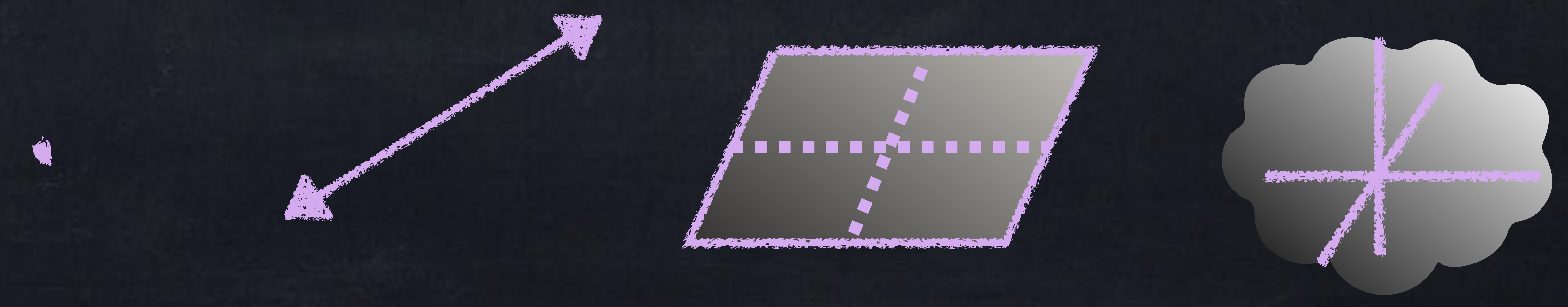
# The Rouché–Capelli Theorem

The system $A\vec{x} = \vec{b}$ has at least one solution if and only if $\operatorname{rank}(A) = \operatorname{rank}\big([A \;\; \vec{b}]\big)$.

Reminder: $A$ is the "coefficient matrix", and $[A \;\; \vec{b}]$ is the "augmented matrix".

If there are any solutions, the collection of all solutions has dimension $n - \operatorname{rank}(A)$, where $n$ is the number of variables.

Dimension: 0 1 2 3



The matrix

$$A = \begin{bmatrix} 5 & 2 & 7 \\ 1 & 0 & 1 \\ 12 & 7 & 19 \end{bmatrix}$$

has determinant $0$ (and rank $2$, which we calculated last week).

What does that mean for $A\vec{x} = \vec{b}$ ?

# Rank/system examples

Last time

Ex 2.

$$\begin{cases} 5x + 2y + 7z = 6 \\ x + z = 4 \\ 12x + 7y + 19z = 9 \end{cases}$$

$$A = \begin{bmatrix} 5 & 2 & 7 \\ 1 & 0 & 1 \\ 12 & 7 & 19 \end{bmatrix}$$

$$[A \ \vec{b}] = \left[\begin{array}{ccc|c} 5 & 2 & 7 & 6 \\ 1 & 0 & 1 & 4 \\ 12 & 7 & 19 & 9 \end{array}\right]$$

$\text{rank}(A) = 2$

$\text{rank}(A|b) = 3$

The coefficient and augmented matrices have different ranks, so there are no solutions to the system.

# Rank/system examples



Ex 3.

$$\begin{cases} 5x + 2y + 7z = \underline{10} \\ x + z = \underline{4} \\ 12x + 7y + 19z = \underline{13} \end{cases}$$

$$A = \begin{bmatrix} 5 & 2 & 7 \\ 1 & 0 & 1 \\ 12 & 7 & 19 \end{bmatrix}$$

$$[A \ \vec{b}] = \left[\begin{array}{ccc|c} 5 & 2 & 7 & \underline{10} \\ 1 & 0 & 1 & \underline{4} \\ 12 & 7 & 19 & \underline{13} \end{array}\right]$$

rank(A) = 2

rank(A|b) = 2

The coeff. and augmented matrices have the same rank, so the system does have at least one solution. The space of all solutions has dimension

(# of variables) − (rank of A) = 3 − 2 = 1,

so the set of solutions is a LINE in 3D space.

# Free variables

How can we describe the solutions nicely when there are infinitely many?

A **free variable** is a variable whose value can be set to anything when describing solutions to a system.

- If $n - \mathrm{rank}(A) = d$, we have $d$ free variables.
- We can choose which of the variables are free.

# Free variables

Ex 3 again

$$\begin{cases} 5x + 2y + 7z = 10 \\ x + z = 4 \\ 12x + 7y + 19z = 13 \end{cases}$$

rank(A) = 2

rank(A|b) = 2

(# of vars.)−rank(A) = 1

We know we have exactly one free variable.
We can pick any one of x or y or z for that variable.

With x free, all solutions look like (x, x−9, 4−x)

With y free: (x,y,z) = (y+9, y, −y−5)

With z free: (x,y,z) = (4−z, −5−z, z)

# Rank and determinant

$$\begin{cases} 5x + 2y + 7z = 6 \\ x + z = 4 \\ 12x + 7y + 18z = 9 \end{cases}$$

$$A = \begin{bmatrix} 5 & 2 & 7 \\ 1 & 0 & 1 \\ 12 & 7 & 18 \end{bmatrix}$$

$\det(A) \neq 0$

$\mathrm{rank}(A) = 3$

$\text{n-rank}(A) = 0$

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$$\begin{cases} 5x + 2y + 7z = 6 \\ x + z = 4 \\ 12x + 7y + 19z = 9 \end{cases}$$

$$\begin{cases} 5x + 2y + 7z = 10 \\ x + z = 4 \\ 12x + 7y + 19z = 13 \end{cases}$$

$$A = \begin{bmatrix} 5 & 2 & 7 \\ 1 & 0 & 1 \\ 12 & 7 & 19 \end{bmatrix}$$

$\det(A) = 0$

$\mathrm{rank}(A) = 2$

For an $n \times n$ matrix $A$, $\det(A) = 0$ if and only if $\mathrm{rank}(A) < n$.

# Right-hand side zeros

What can we say about a square system $A\vec{x} = \vec{0}$ where the right-side has only zeros?

$$\begin{cases} 5x + \phantom{2y +{}} z = 0 \\ 2x + 2y + 3z = 0 \\ -8x + 2y + \phantom{3}z = 0 \end{cases}$$

- $(x, y, z) = (0,0,0)$ is a solution.
- In order to have any *other* solutions, the coefficient matrix must have a determinant of $0$ (because if not then we could solve $\vec{x} = A^{-1}\vec{0} = \vec{0}$).
  - In that case there will be infinitely many solutions.
    The set of all solutions will form a line or a plane in 3D space.

Systems of equations appear in many kinds of tasks.

They are not always in the format $\begin{cases} \_x + \_y = \_ \\ \_x + \_y = \_ \end{cases}$.

Task: Describe *all* vectors $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ for which $\begin{bmatrix} 6 & 4 \\ 1 & 3 \end{bmatrix} \vec{v} = 7\,\vec{v}$.

Using y as a free variable, $\begin{bmatrix} 4y \\ y \end{bmatrix}$ for any y.

In other words, these are all the multiples of [4,1].

# Eigenvectors and eigenvalues

For a square matrix $A$, if we have

$$A\vec{v} = s\vec{v}$$

for some number $s$ and some vector $\vec{v} \neq \vec{0}$ then

- the vector $\vec{v}$ is called an **eigenvector** of $A$, and
- the number $s$ is called an **eigenvalue** of $A$.

Note that if $\vec{v}$ is an eigenvector, any scalar multiple of $\vec{v}$ will also be an eigenvector.

# Eigenvectors and eigenvalues

We just saw that

$$\begin{bmatrix} 6 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 7 \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

Using this new vocabulary, we can say that

- $[4,1]$ is an eigenvector of $\begin{bmatrix} 6 & 4 \\ 1 & 3 \end{bmatrix}$.
- 7 is an eigenvalue of $\begin{bmatrix} 6 & 4 \\ 1 & 3 \end{bmatrix}$.

If you know an eigenvalue of a matrix, the method we already used is how you find eigenvectors.

But how do you find eigenvalues?

Example: Find the eigenvalues of $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

**Warm-up:**

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - p\, I_{2\times 2}$$

$\det(A - pI) = 0$ if $p = -1$, $p = 4$

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 1-s & 2 \\ 3 & 2-s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $\det\begin{pmatrix} 1-s & 2 \\ 3 & 2-s \end{pmatrix} \neq 0$ then this has exactly 1 solution (which will be $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$). If $\det\begin{pmatrix} 1-s & 2 \\ 3 & 2-s \end{pmatrix} = 0$ then this has $\infty$ solutions.

We want

$$(1-s)(2-s) - (2)(3) = 0.$$

So the eigenvalues are $s = -1$ and $s = 4$.

# Finding eigenvalues

The eigenvalues of $A$ are the values of $s$ for which $\det(A - sI) = 0$.

Proof: if $A\vec{v} = s\vec{v}$ and $\vec{v} \neq \vec{0}$ then

$$A\vec{v} = I\,(s\vec{v})$$

$$A\vec{v} - Is\vec{v} = \vec{0}$$

$$(A - Is)\vec{v} = \vec{0} \quad \text{with } \vec{v} \neq \vec{0}$$

$$\det(A - Is) = 0$$

# Finding eigenvalues

The eigenvalues of $A$ are the values of $\lambda$ for which $\det(A - \lambda I) = 0$.

In most book/websites, the Greek lowercase lambda $\lambda$ is used for eigenvalues.

Determinants and eigenvalues are also related in the following way:

If $\lambda_1, ..., \lambda_n$ are the eigenvalues of $A$ (counted with algebraic multiplicity*), then

$$\det(A) = \lambda_1 \times \lambda_2 \times \cdots \times \lambda_n.$$

* We will define this in January.

For an $n \times n$ matrix $A$, either…

ALL of these are true:

- $A$ is invertible
- $\det(A) \neq 0$
- $0$ is not an eigenvalue
- $\text{rank}(A) = n$

or

ALL of these are true:

- $A$ is non-invertible
- $\det(A) = 0$
- $0$ is an eigenvalue
- $\text{rank}(A) < n$

Task: Find the eigenvalues of $A = \begin{bmatrix} 3 & 10 \\ 1 & 5 \end{bmatrix}$.

Solving $(3 - \lambda)(5 - \lambda) - (10)(1) = 0$

gives $\lambda_1 = 4+\sqrt{11}$ and $\lambda_2 = 4-\sqrt{11}$.

Task: Find the eigenvalues of $A = \begin{bmatrix} 2 & -10 \\ 1 & 8 \end{bmatrix}$.

$$\det(A) = 0$$
$$(2-s)(8-s) + 10 = 0$$
$$s^2 - 10s + 26 = 0$$

$$\Delta = (-10)^2 - 4(1)(26)$$
$$= 100 - 104$$
$$= -4$$

$$s = (10 \pm \sqrt{-4})/2$$ 🧐

What does $\sqrt{-4}$ mean?

Next week: complex numbers!