Math 1433

18 December 2023

Warm-up:

Expand $(5 - x)(2 + 4x)$.

Eigen...

For a square matrix A, if we have

$$A\vec{v} = \lambda\vec{v}$$

for some number λ and some vector $\vec{v} \neq \vec{0}$ then

- the vector $\vec{v}$ is called an **eigenvector** of A, and
- the number λ is called an **eigenvalue** of A.

The letter λ is a lowercase Greek “lambda”.

Note that if $\vec{v}$ is an eigenvector, any scalar multiple of $\vec{v}$ will also be an eigenvector.

Finding eigenvalues

Last time

The eigenvalues of A are the values of λ for which $\det(A - \lambda I) = 0$.

Proof: if $A\vec{v} = \lambda\vec{v}$ and $\vec{v} \neq \vec{0}$ then

$$A\vec{v} = I(\lambda\vec{v})$$

$$A\vec{v} - \lambda I\vec{v} = \vec{0}$$

$$(A - \lambda I)\vec{v} = \vec{0} \quad \text{with } \vec{v} \neq \vec{0}$$

$$\det(A - \lambda I) = 0$$

Last time

If $\lambda_1, ..., \lambda_n$ are the eigenvalues of A (counted with algebraic multiplicity*), then

$$\det(A) = \lambda_1 \times \lambda_2 \times \cdots \times \lambda_n.$$

For an $n \times n$ matrix A, either…

ALL of these are true:

- A is invertible
- $\det(A) \neq 0$
- 0 is not an eigenvalue
- $\operatorname{rank}(A) = n$

or

ALL of these are true:

- A is non-invertible
- $\det(A) = 0$
- 0 is an eigenvalue
- $\operatorname{rank}(A) < n$

* We will define this in January.

Task: If matrix M satisfies

$$M\begin{bmatrix}2\\7\end{bmatrix}=\begin{bmatrix}10\\35\end{bmatrix} \quad \text{and} \quad M\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}-2\\2\end{bmatrix}$$

- give two eigenvalues of M.
- give five eigenvectors of M.
- calculate $\det(M)$.

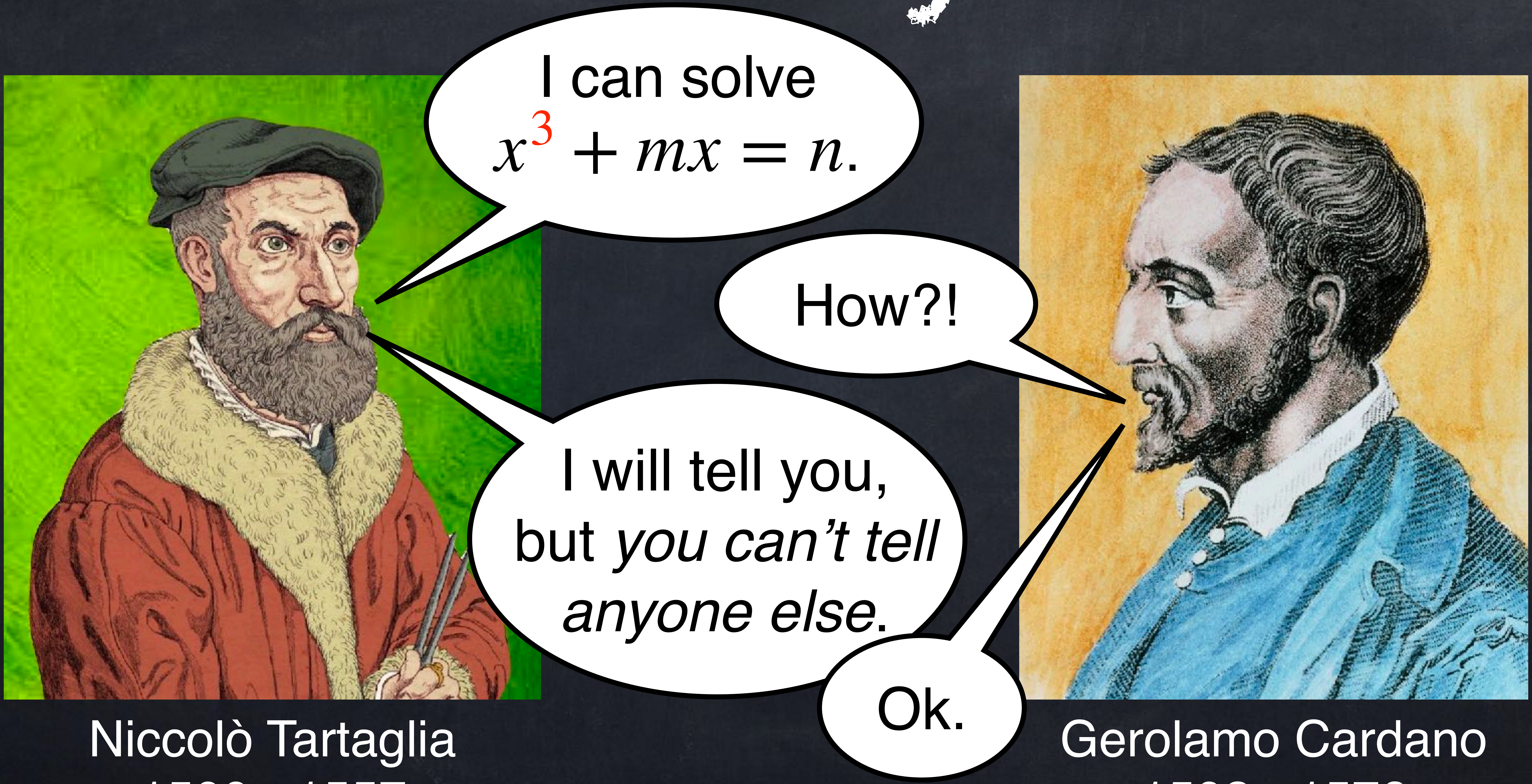
Task: Find the eigenvalues of $A = \begin{bmatrix} 3 & 10 \\ 1 & 5 \end{bmatrix}$.

$\lambda = (8 \pm \sqrt{44})/2 = 4 \pm \sqrt{11}$

Task: Find the eigenvalues of $A = \begin{bmatrix} 2 & -10 \\ 1 & 8 \end{bmatrix}$.

$\lambda = (10 \pm \sqrt{-4})/2$ 🧐

History



Niccolò Tartaglia
1500 - 1557

Gerolamo Cardano
1502 - 1576

Art: Matteo Bergamelli, Science Photo Library
See https://www.youtube.com/watch?v=cUzklzVXJwo for much more detail.

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Ars Magna
(1545)

Art: Matteo Bergamelli, Science Photo Library
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Multiplication

What does 5×3 mean?

- 
- More advanced: no pictures, just $5 + 5 + 5$.

What does $5 \times \frac{1}{3}$ mean? 5.1×9.26 ? $7.4 \times (-12.38)$?

You have changed how you think about multiplication many times already!

"$\sqrt{7}$" is a symbol we use to describe the number for which $\_\_ \times \_\_ = 7$.

Multiplication

From now on, we will say that

$$i \times i = -1.$$

There are many good reasons for this, but for now just consider it a new part of the definition of how multiplication works.

People often write "$i = \sqrt{-1}$".

Algebra with complex #s

Using $i^2 = -1$ and standard algebra rules, we can can now do lots of computations with "complex numbers".

$$5(3+7) = (5\cdot 3) + (5\cdot 7)$$

$$x(3+2x) = x\cdot 3 \ + \ x\cdot 2x$$
$$= 3x + 2x^2$$

$$i(3+2i) = i\cdot 3 \ + \ i\cdot 2i$$
$$= 3i + 2i^2$$
$$= -2 + 3i$$

Algebra with complex #s

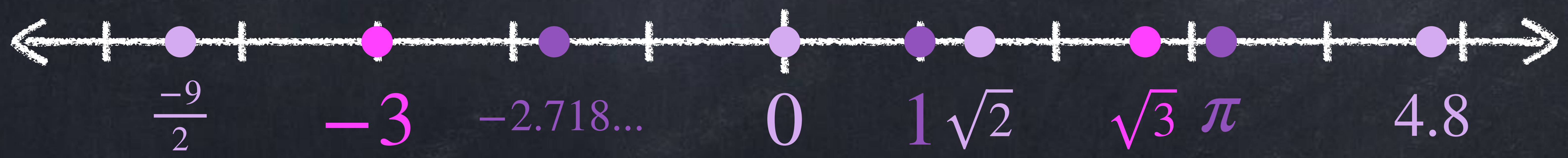
Using $i^2 = -1$ and standard algebra rules, we can can now do lots of computations with “complex numbers”.

The word “complex” here does *not* mean difficult or complicated (skomplikowana).
It means **made-of-multiple-parts** (**zespolona**).

$$14 + 18i$$

Types of numbers

- **Natural** numbers: $0, 1, 2, 3, 4, \ldots$ (in some books, only $1, 2, 3, 4, \ldots$)
- **Integers**: $\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$
- **Rational** numbers are all the numbers that *can* be written as one integer divided by another. Examples: $\frac{1}{2}, \frac{-2}{3}, 1.5, \frac{8}{1} = 8, 0, \frac{-5}{4}$
- **Real** numbers are all the values on a number line. Examples:

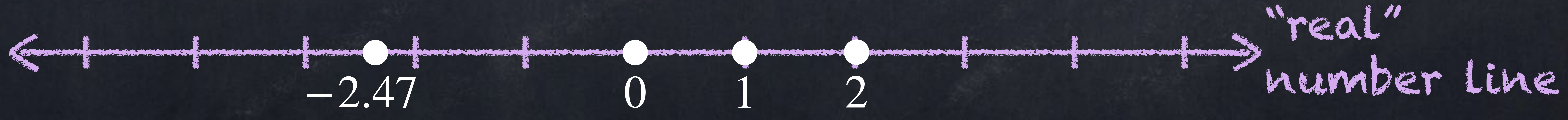


- **Complex** numbers can be written as a real number plus $\sqrt{-1}$ times a real number. Examples: $3+2i, 9.7, \frac{1}{2}-i, \sqrt{-5}$

Complex numbers

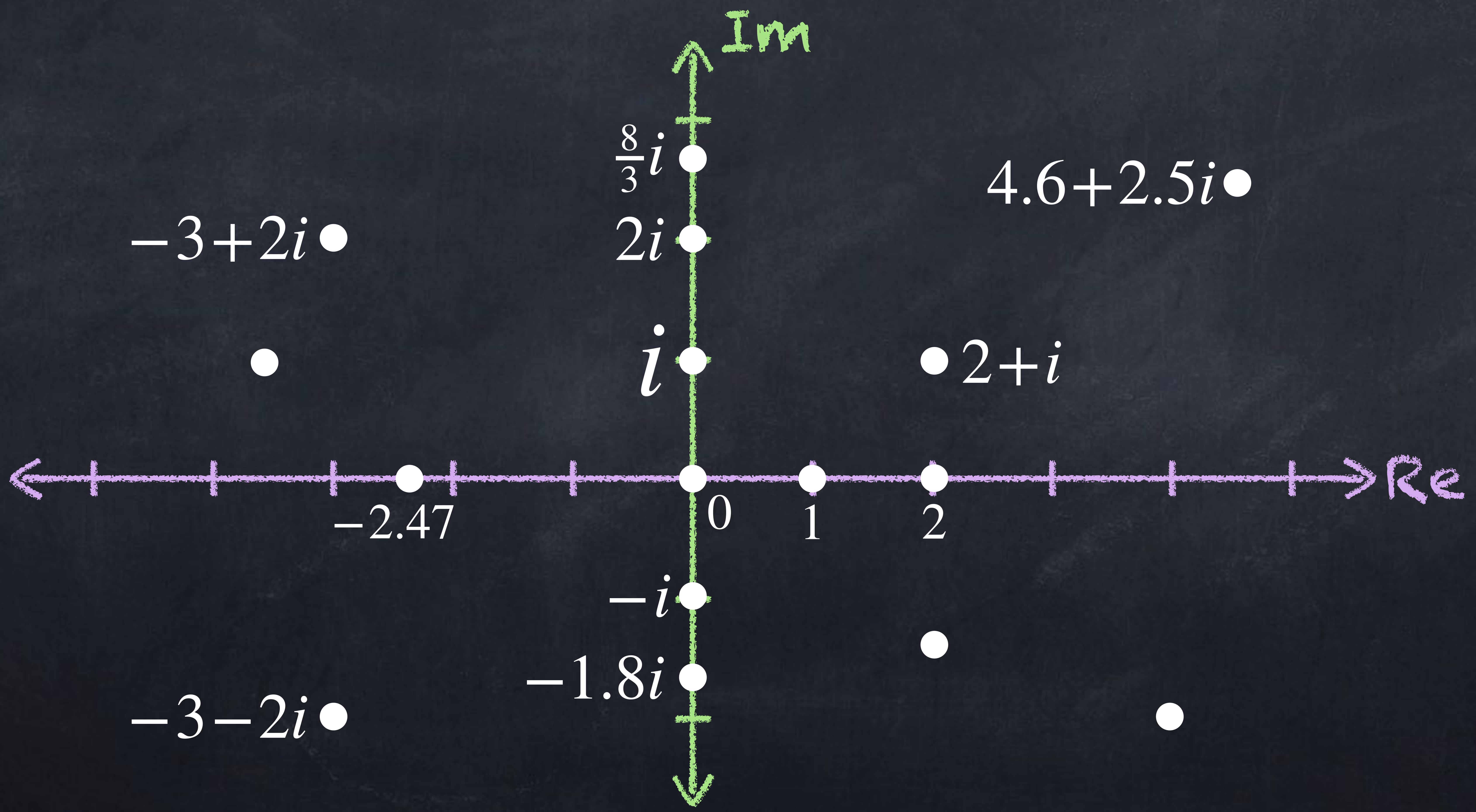
Algebra idea: allow square roots of negative numbers

Geometry idea: 2D number plane

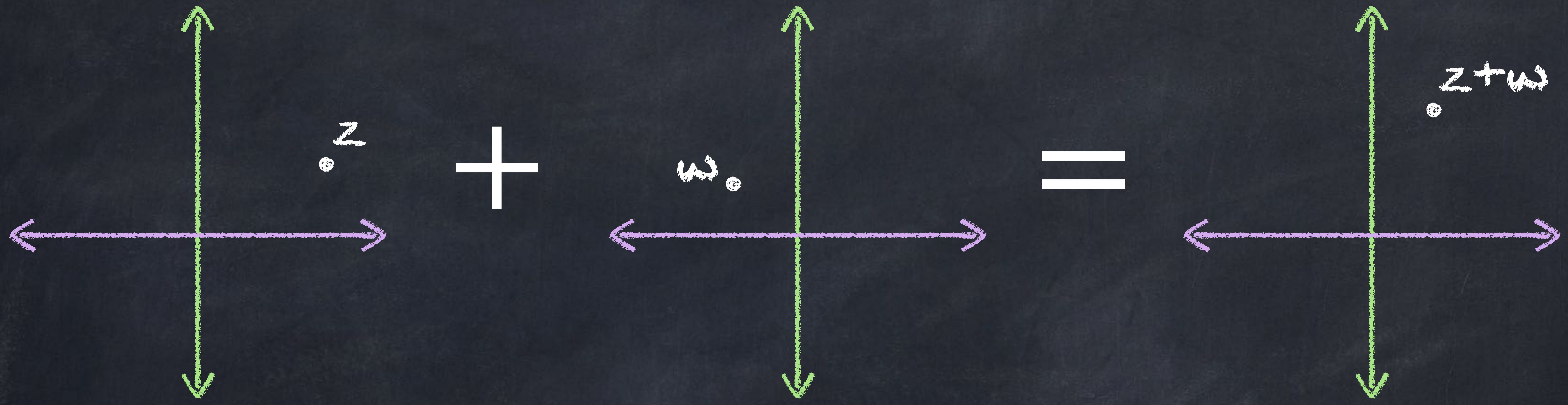


Complex #s in *algebra*: allow square roots of negative numbers.

Complex #s in *geometry*: instead of “number line”, use 2D number plane!

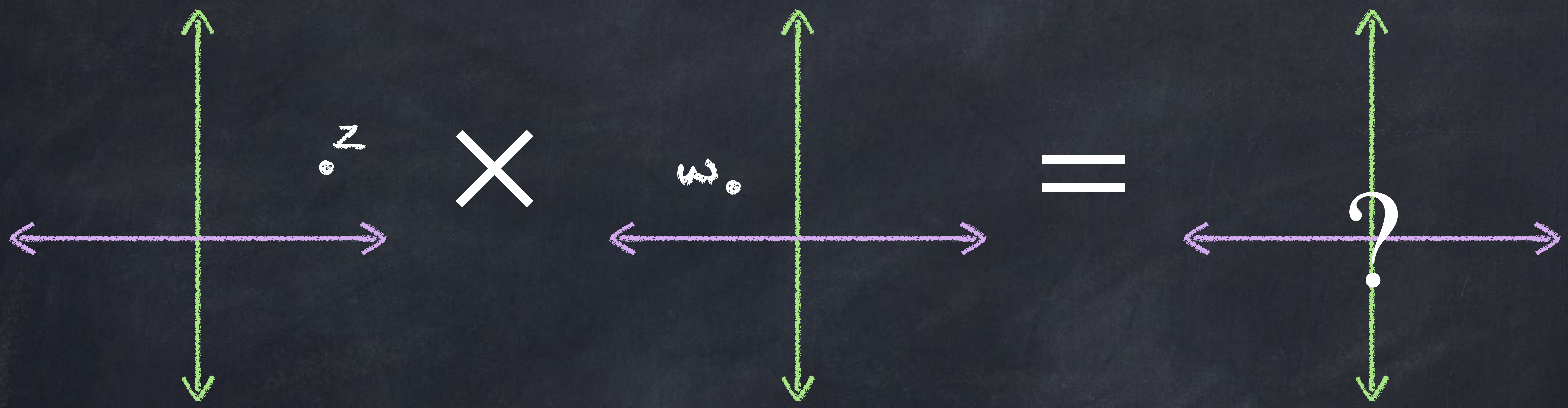


What goes it mean *geometrically* to **add** complex numbers?



Addition of complex numbers works basically the same as addition of 2D vectors!

What goes it mean *geometrically* to **multiply** complex numbers?



This is more difficult than addition. We will discuss it in the next lecture.

- Special case we can easily say now: if r is a real number then $r\,z$ works just like “scalar multiplication” $r\vec{v}$ with a vector.

Rectangular form

We call the horizontal (left/right) part of a complex number its **real part**, and we call the vertical (up/down) part its **imaginary part**.

Example:

- The real part of $7 + 2i$ is just 7.
- The imaginary part of $7 + 2i$ is just 2.
  - Note: it is <u>not</u> $2i$.
  - The "imaginary part" is actually a real number.

Rectangular form

We call the horizontal (left/right) part of a complex number its **real part**, and we call the vertical (up/down) part its **imaginary part**.

Writing a complex number as

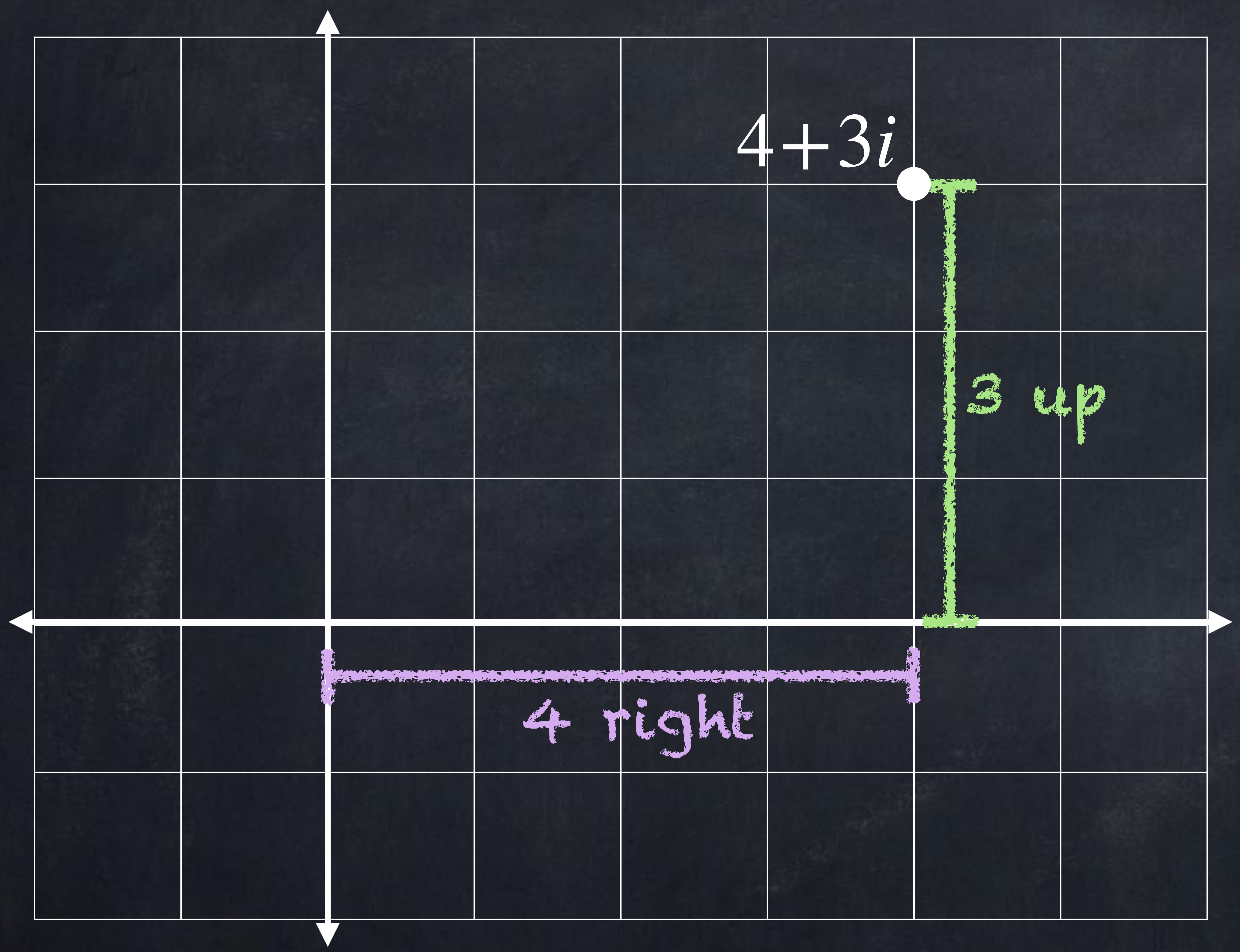
$$\_\_\_\_ + \_\_\_\_ i,$$

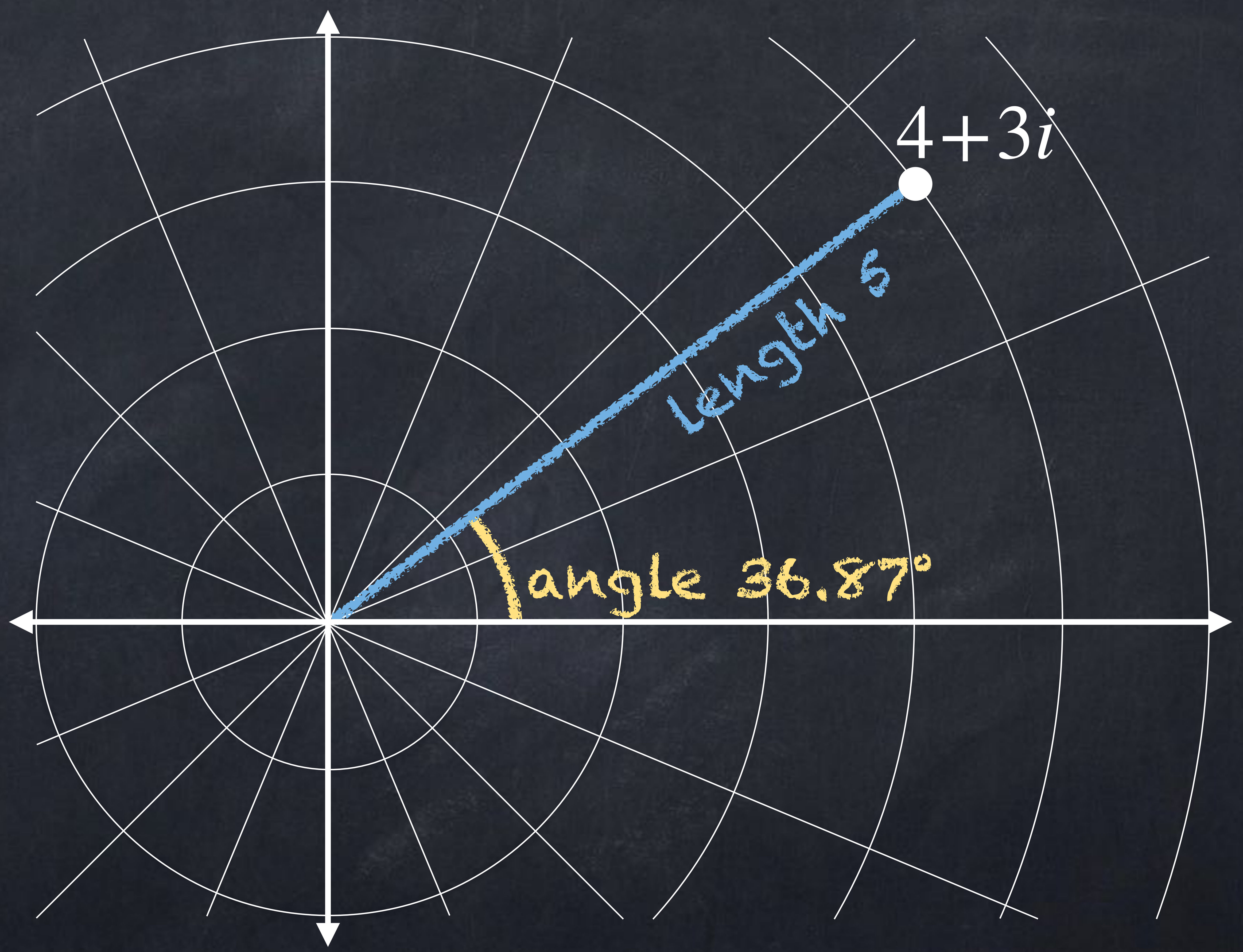
where both blanks are real numbers, is called **rectangular form**. If one of the blanks is 0, we can skip that part and still call it rectangular form.

Examples:

- $(1+i)^3$ in rectangular form is $-2+2i$.
- i^3 in rectangular form is $-i$.

Polar form





Instead of 4 right and 3 up, we can describe this point as being 5 units away at an angle of 36.87°.

Magnitude and argument

The **magnitude**[1] of a complex number is its *distance* from 0.

We write z for the magnitude of a complex number z.

Examples:

- The magnitude of $4+3i$ is 5.
  - In symbols, this is written “ $4+3i \quad = 5$ ”.
- $2-7i \quad = \sqrt{53}$
- $-8 \quad = 8$
- $a+bi \quad = \sqrt{a^2+b^2}$ if a and b are real

1. This is also called modulus, or norm, or absolute value.

Magnitude and argument

The **argument** of a complex number is the angle between the positive real axis and the line from 0 to that complex number.

We write $\arg(z)$ for the argument (the angle) of a complex number z.

Examples:

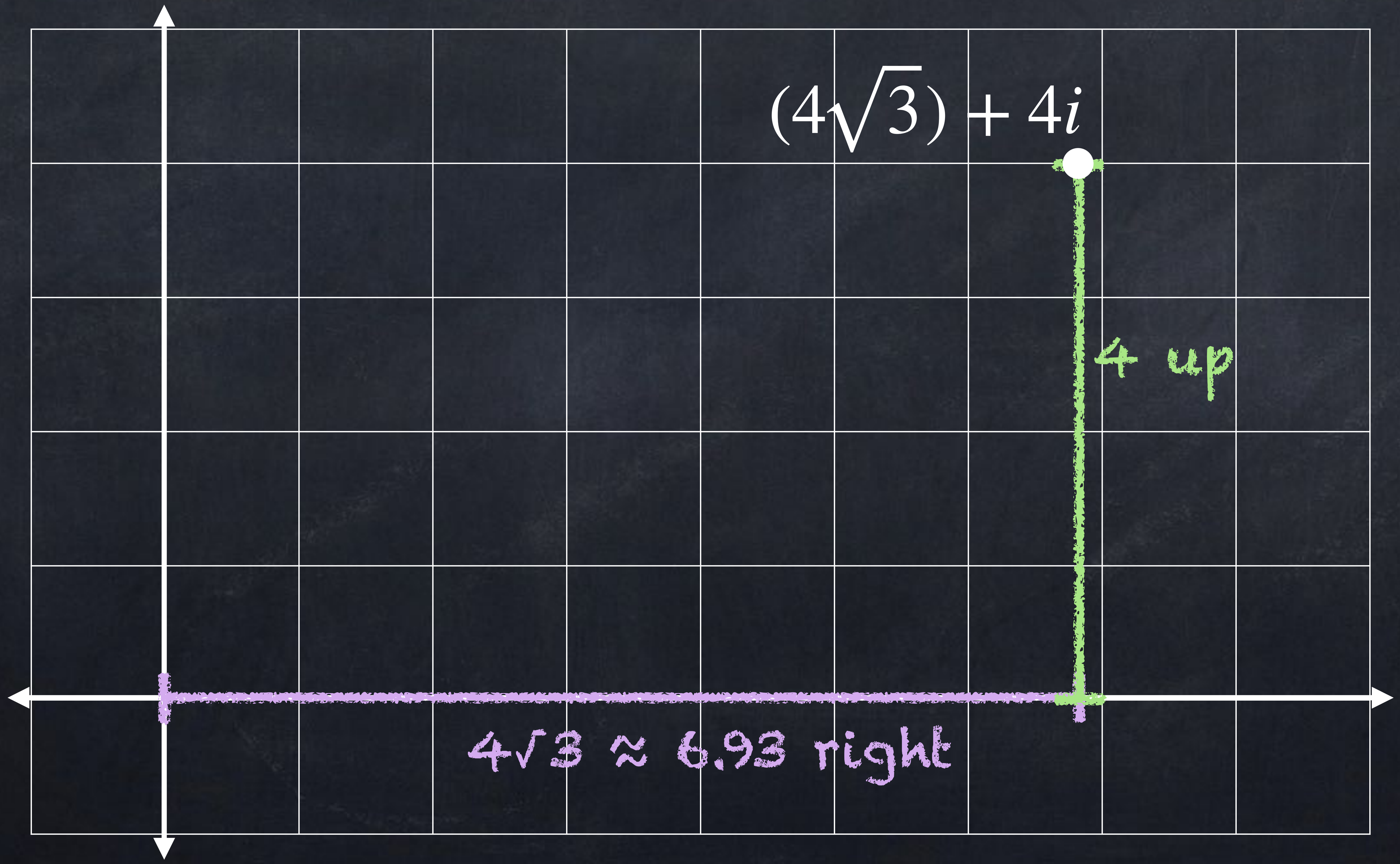
- The argument of $1+i$ is $45°$.
  - In symbols, " $\arg(1+i) = 45°$ ".
- $\arg(\sqrt{3}+i) = \frac{\pi}{6}$.
- The argument of $4+3i$ is $\arctan(\frac{3}{4})$, also written $\operatorname{atan}(\frac{3}{4})$ or $\tan^{-1}(\frac{3}{4})$. A calculator can tell us this is approximately 0.6435, or $36.89°$.

Where is the point

$$z = 8\cos(\pi/6) + 8\sin(\pi/6)\, i$$

on the complex plane?

$z = 8(\sqrt{3}/2) + 8(1/2)i$



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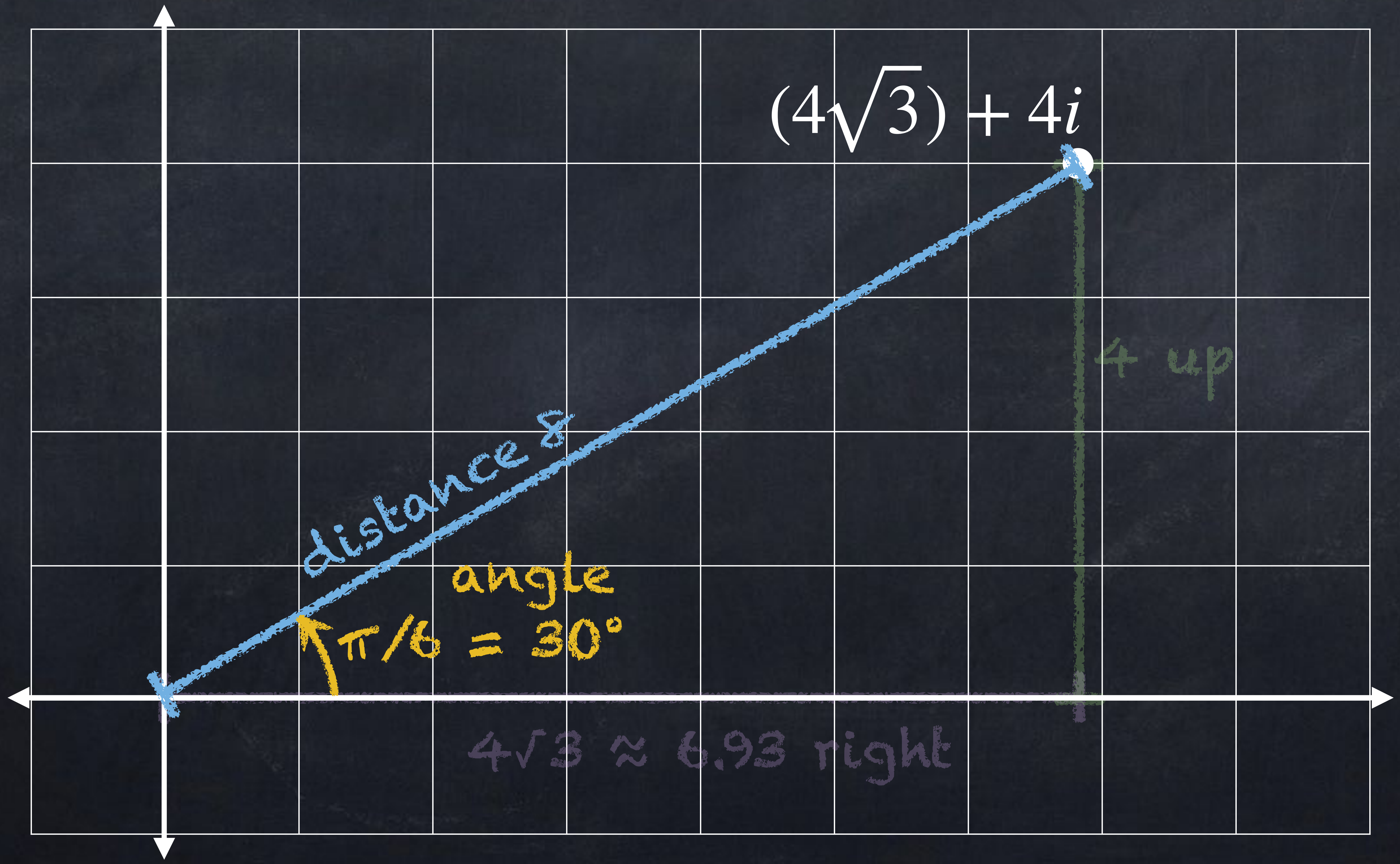


For a number

$$z = a + bi,$$

it can be difficult to find the magnitude and argument.

- This way of writing complex numbers is called **rectangular form**.

must be same number

must be same number

For a number

$$z = r\cos(\theta) + r\sin(\theta)\, i$$

with $r > 0$, the magnitude is exactly r and the argument is exactly θ.

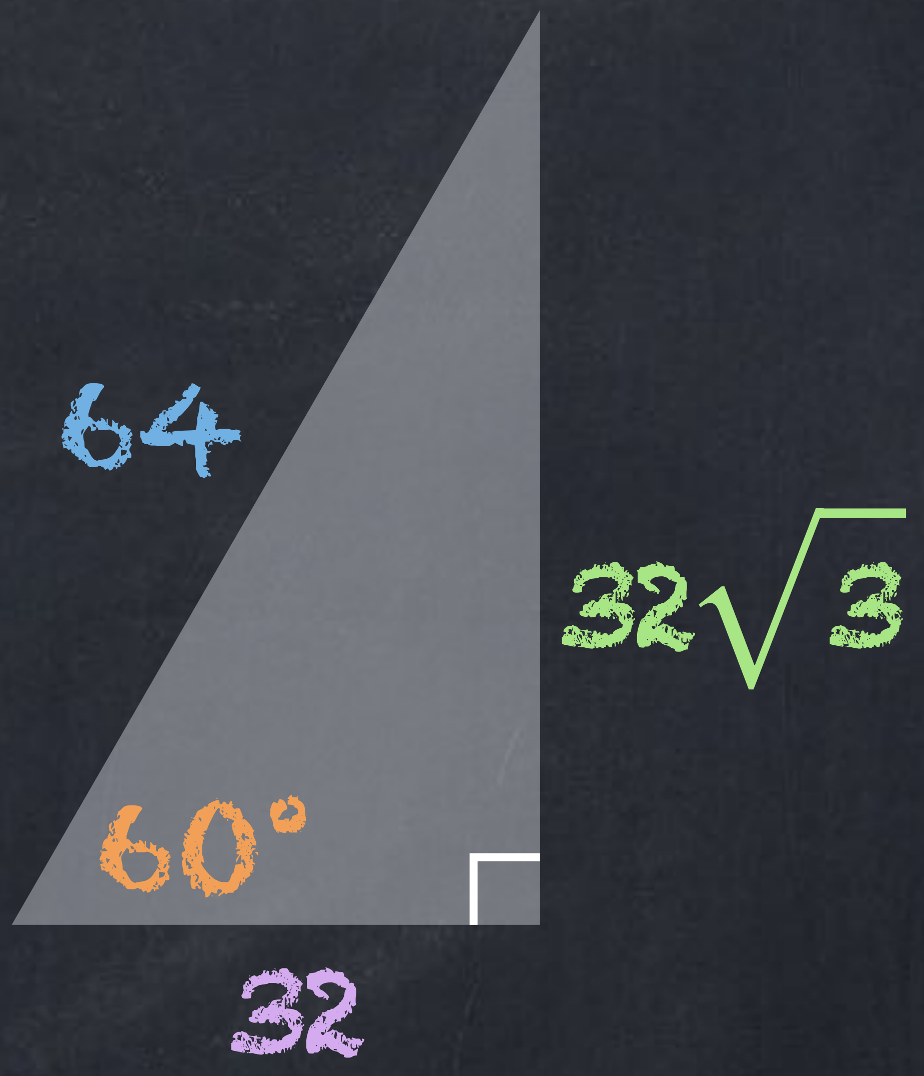
- This way of writing complex numbers is called **polar form**.

Task: calculate z^2 for

$$z = 8\cos(30°) + 8\sin(30°)\,i,$$

giving the answer in both rectangular *and* polar forms.

$$z^2 = (4\sqrt{3} + 4i)(4\sqrt{3} + 4i)$$
$$= 4\cdot 4\cdot 3 + 16\sqrt{3}i + 16\sqrt{3}i + 16i^2$$
$$= 48 + 32\sqrt{3}i - 16$$
$$= \boxed{32 + 32\sqrt{3}i} \leftarrow \text{rectangular form}$$



Because $|z^2| = \sqrt{32^2 + (32\sqrt{3})^2} = 64$

and $\arg(z^2) = 60°$ or $\pi/3$, we have that

$$z^2 = \boxed{64\cos(60°) + 64\sin(60°)i} \leftarrow \text{polar form}$$