Math 1433

8 January 2024

Warm-up 1:
Simplify $(8^{1/2})^6$.

Warm-up 2:
Give an angle that has $\cos(\theta) = \frac{1}{\sqrt{2}}$ and $\sin(\theta) = -\frac{1}{\sqrt{2}}$.

Complex numbers

Last time

You should already be able to expand

$$(5 - x)(2 + 4x) = \cdots = 10 + 18x - 4x^2.$$

If we define i as a number for which

$$i^2 = -1,$$

we can calculate, for example,

$$(5 - i)(2 + 4i) = \cdots = 10 + 18i - 4i^2$$
$$= 10 + 18i + 4 = 14 + 18i.$$

Last time

A complex number is anything that can be written as

$$a + bi,$$

where a and b are both real numbers (either or both can be zero).

- The **real part** is the number a.
- The **imaginary part** is the number b (which is a real number).

The word "complex" here does *not* mean difficult or complicated (skomplikowana).
It means **made-of-multiple-parts** (**zespolona**).

Magnitude and argument

The **magnitude**[1] of a complex number is its *distance* from 0.

We write z for the magnitude of a complex number z.

Examples:

- The magnitude of $4+3i$ is 5.
    - In symbols, this is written " $4+3i \quad = 5$ ".
- $2-7i \quad = \sqrt{53}$
- $-8 \quad = 8$
- $a+bi \quad = \sqrt{a^2+b^2}$ if a and b are real

1. This is also called modulus, or norm, or absolute value.

Magnitude and argument

The **argument** of a complex number is the angle between the positive real axis and the line from 0 to that complex number.

We write $\arg(z)$ for the argument (the angle) of a complex number z.

Examples:

- The argument of $1+i$ is $45°$.
  - In symbols, " $\arg(1+i) = 45°$ ".
- $\arg(\sqrt{3}+i) = \frac{\pi}{6}$.
- The argument of $4+3i$ is $\arctan(\frac{3}{4})$, also written $\operatorname{atan}(\frac{3}{4})$ or $\tan^{-1}(\frac{3}{4})$.

  A calculator can tell us this is approximately 0.6435, or $36.89°$.

Last time

For a number

$$z = a + bi,$$

it can be difficult to find the magnitude and argument.

- This way of writing complex numbers is called **rectangular form**.

For a number

must be same number

$$z = r\cos(\theta) + r\sin(\theta)\, i$$

must be same number

with $r > 0$, the magnitude is exactly r and the argument is exactly θ.

- This way of writing complex numbers is called **polar form**.

Write

$$8\sqrt{3} - \sqrt{147} + \sqrt{-3}$$

in both

- rectangular form $z = a + bi$.

$$\sqrt{3} + \sqrt{3}\,i$$

- polar form $z = r\cos(\theta) + r\sin(\theta)\,i$.

$$\sqrt{6}\cos(45^\circ) + \sqrt{6}\sin(45^\circ)i$$

Quiz 5

is moved to next week.

If

$$z = 8\cos(\pi/6) + 8\sin(\pi/6)\, i,$$

calculate z^2, giving your answer in polar form.

$$z^2 = ((4\sqrt{3}) + 4i)^2$$
$$= (4\sqrt{3})^2 + 2(4\sqrt{3})(4i) + (4i)^2$$
$$= (48 - 16) + (32\sqrt{3})i$$
$$= 32 + (32\sqrt{3})i$$
$$= 32(1 + \sqrt{3}\, i)$$
$$= 64\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$$
$$z^2 = 64\cos(\pi/3) + 64\sin(\pi/3)\, i$$

For the previous example, we saw

$$\big(r\cos(\theta) + r\sin(\theta)\,i\big)^2 = r^2\cos(2\theta) + r^2\sin(2\theta)i.$$

In fact,

$$\big(r\cos(\theta) + r\sin(\theta)\,i\big)^n = r^n\cos(n\theta) + r^n\sin(n\theta)i$$

is true for any n, including fractions and negative values.

You do *not* have to memorize this fact, though. Instead, we will use a shortcut—which you *will* need to memorize—that will make these formulas obvious.

Exponential form

FACT: $e^{\theta i} = \cos(\theta) + \sin(\theta)\, i$.

I am *not* going to explain why this is true. Instead, just think about this:

- Remember that $3 \cdot x$ means $x + x + x$, but $-4.73 \cdot x$ is *not* about repeated addition. With matrices, AB or $A\vec{v}$ can be thought of as applying a linear transformation. Still, these these are all multiplication.
- Similarly, x^3 means $x \cdot x \cdot x$, but $x^{-4.73}$ is *not* about repeated multiplication. Still, these are both powers.

$$e^{\theta i} = \cos(\theta) + \sin(\theta)\, i$$

Exponential form

Fact: $e^{\theta i} = \cos(\theta) + \sin(\theta)\, i$

Multiplying both sides of this by r gives

$$r\, e^{\theta i} = r\cos(\theta) + r\sin(\theta)\, i.$$

Writing a complex number as $\_\_ e^{\_\_ i}$ is called **exponential form**.

Example: $4\sqrt{3} + 4i \quad = \quad 8\cos(\frac{\pi}{6}) + 8\sin(\frac{\pi}{6})\, i \quad = \quad 8e^{(\pi/6)i}$

rectangular form

polar form

exponential form

Exponential form

Fact: $e^{\theta i} = \cos(\theta) + \sin(\theta)\, i$

Basic algebra tells us that

$$(r\, e^{\theta i})^n = r^n e^{n\theta i}.$$

Using the fact at the top turns this equation into

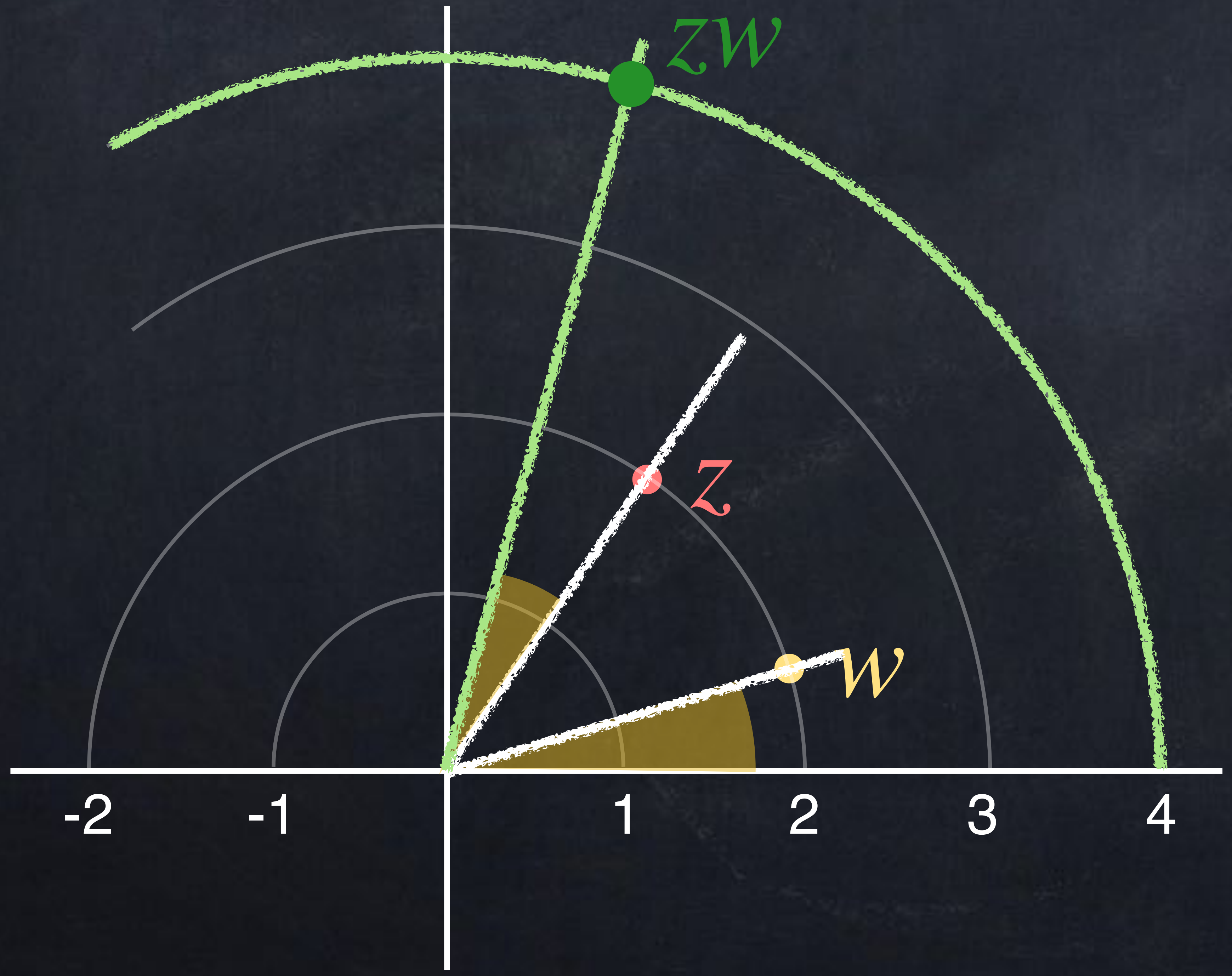
$$\left(r\cos(\theta) + r\sin(\theta)\, i\right)^n = r^n\cos(n\theta) + r^n\sin(n\theta)i.$$

Multiplication

In general, $(re^{\theta i}) \cdot (se^{\phi i}) = (rs)e^{(\theta+\phi)i}$.

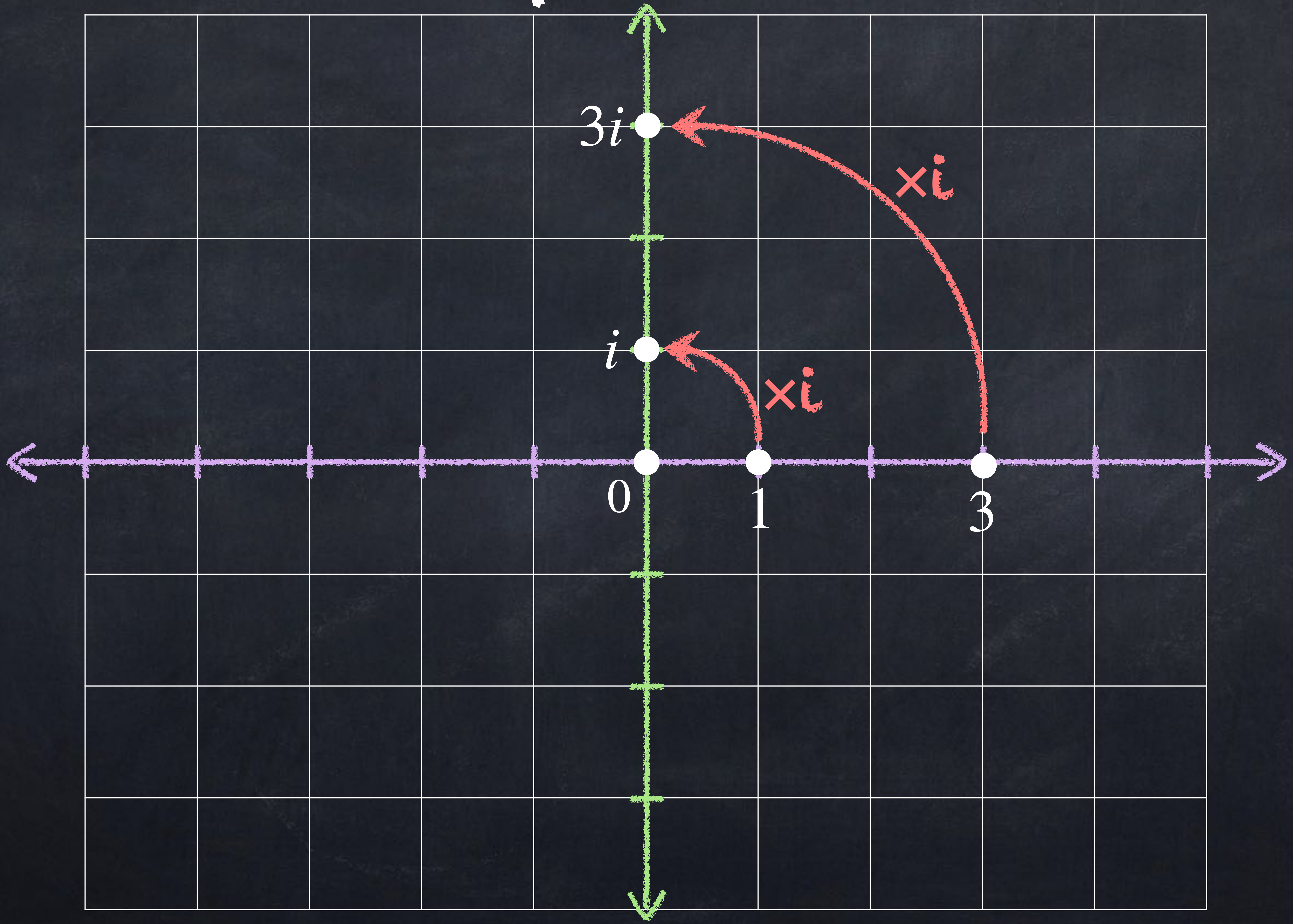
What does this mean *visually*?

Let's draw a dot • at zw below.



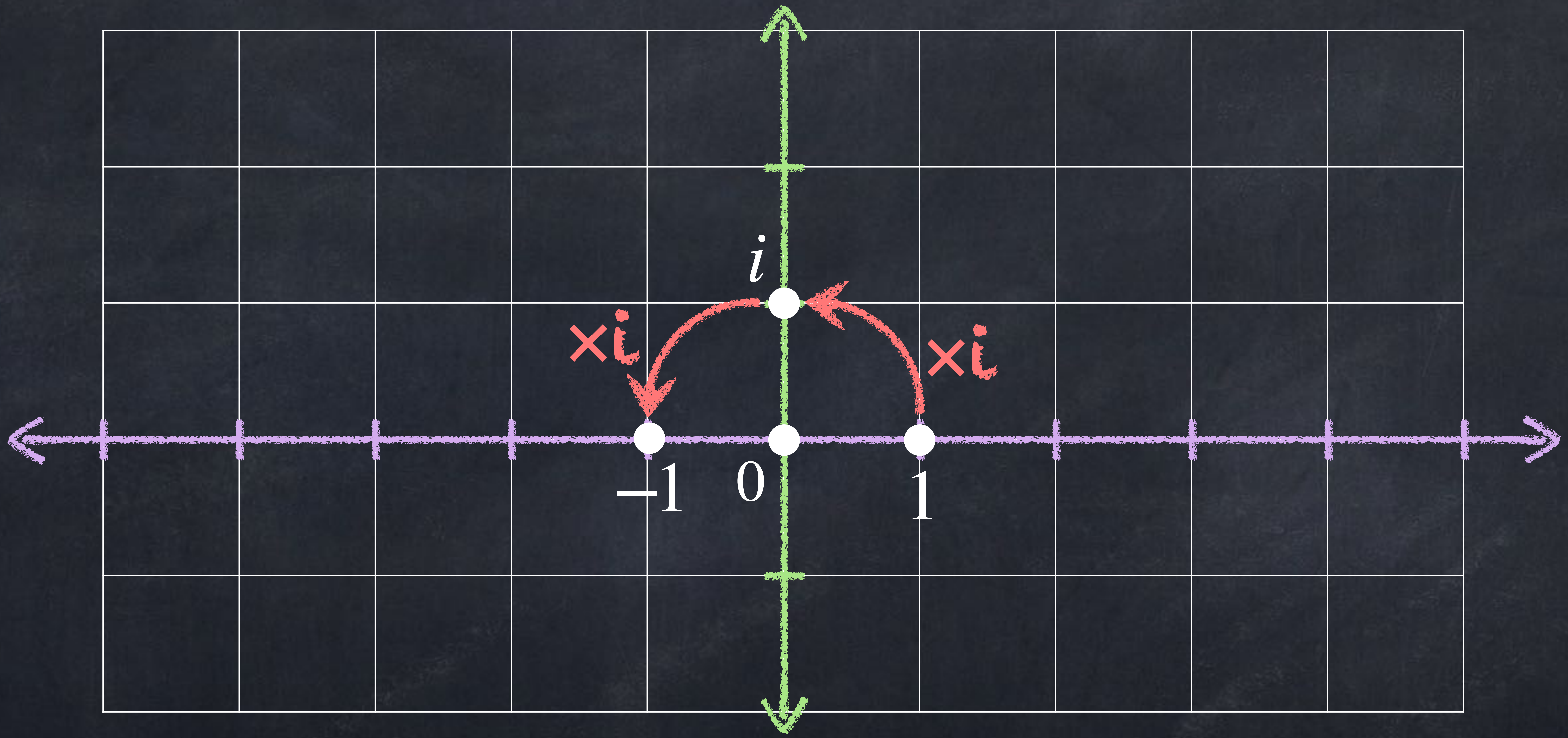
Multiplication

Multiplying by i rotates a complex number by 90°.



What if we rotate by 90° and then rotate by 90° again?

We get $1 \times i \times i = -1$.



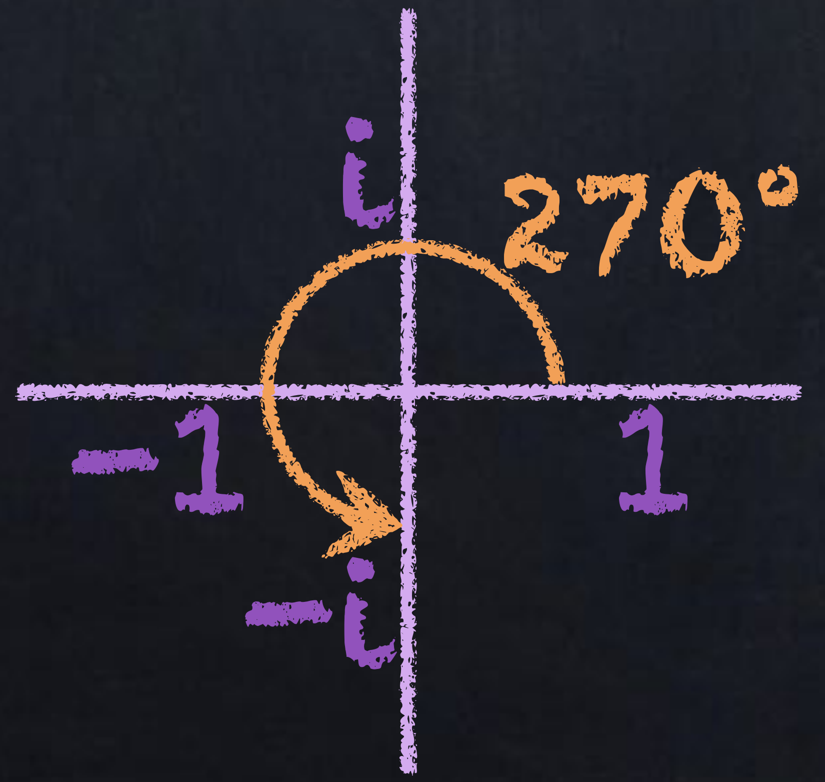
This provides another explanation for why the powers of i repeat the same four values and why $i^{-1} = -i$ (rotating 90° *clock-wise* from 1) and other negative powers simplify the way they do.

Task 1: Write $2+2i$ in exponential form. $\sqrt{8}\, e^{45° i}$

Task 2: Write $(2+2i)^6$ in exponential form.

$$(2+2i)^6 = \left(\sqrt{8}\, e^{45° i}\right)^6$$
$$= (\sqrt{8})^6\, e^{(45° \cdot 6) i}$$
$$= \boxed{512\, e^{270° i}}$$

Task 3: Write $(2+2i)^6$ in rectangular form.



$$e^{270° i} = -i\text{, so } 512 e^{270° i} = \boxed{-512i}$$

Rect. and exponential forms

Multiplying $\left(\frac{-7}{2}+\frac{7\sqrt{3}}{2}i\right)\left(2-2\sqrt{3}\,i\right)$ is possible, but it takes a lot of algebra work.

Multiplying $\left(7e^{\frac{2\pi}{3}i}\right)\left(4e^{\frac{-\pi}{3}i}\right)$ is much easier:

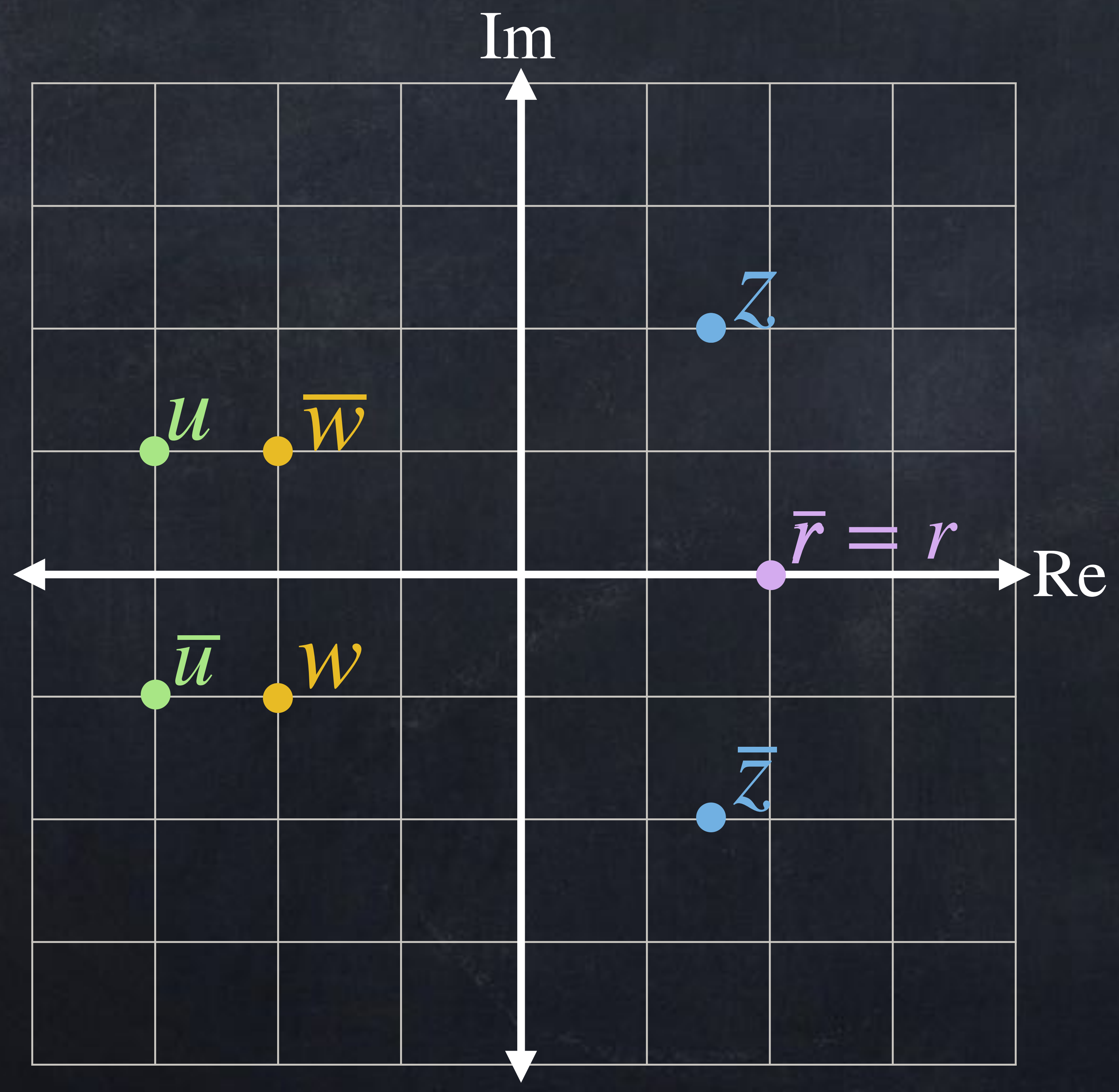
$$\left(7\cdot 4\right)e^{\left(\frac{2\pi}{3}i+\frac{-\pi}{3}i\right)} = 28e^{\left(\frac{\pi}{3}i\right)}.$$

We can then expand this to $28\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=14+14\sqrt{3}\,i$ if we need to.

Complex conjugate

The **complex conjugate** (or just **conjugate**) of a complex number z is the reflection of z across the real axis.

It is written $\overline{z}$ and spoken as "z bar".



Complex conjugate

How is $\overline{z}$ calculated?

$$\overline{a + bi} = a - bi$$

$$\overline{r\,e^{\theta i}} = r\,e^{-\theta i}$$