Math 1433

16 October 2023

Raise you hand if you have used writing like

$$x \in \mathbb{R}$$

before.

Set notation

A **set** is a collection of objects. In this class, we will mostly be interested in collections of vectors (collections of points!).

- For a finite set, you can just list them. Example:

$$S = \left\{ \begin{pmatrix} 3 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}.$$

- For an infinite set, it's often better to use "set-builder" notation. Example:

$$A = \left\{ \begin{pmatrix} 5 \\ y \end{pmatrix} : y \geq 0 \right\}.$$

Set notation

Some specific sets have their own symbols:

$\mathbb{N}$ is the set of all natural numbers.

$\mathbb{R}$ is the set of all real numbers.

$\mathbb{R}^2$ is the set of all points on the xy-plane *OR* the set of all 2D vectors.

$\mathbb{R}^3$ is the set of all points in 3D space *OR* the set of all 3D vectors.

The symbol $\in$ is used to show that an objects belongs in a set:

$$5 \in \mathbb{N}$$

$$5 \in \mathbb{R}$$

$$[2, 10.1] \in \mathbb{R}^2$$

$$\begin{bmatrix} 1 \\ \pi \\ \sqrt{2} \end{bmatrix} \in \mathbb{R}^3$$

$$[0,0,0] \in \mathbb{R}^3$$

Linear combinations

A **linear combination** of some vectors is any sum of scalar multiples of those vectors.

- In symbols, $\vec{u}$ is a linear combination of $\vec{v}$ and $\overrightarrow{w}$ if
$$\vec{u} = a\vec{v} + b\overrightarrow{w}$$
for some numbers a and b.

- For more vectors, $\vec{u}$ is a linear combination of $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n}$ if
$$\vec{u} = s_1\overrightarrow{v_1} + s_2\overrightarrow{v_2} + \cdots + s_n\overrightarrow{v_n}$$
for some numbers $s_1, \ldots, s_n$.

Linear combinations

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for some numbers a and b.

Example 1: Write $\begin{bmatrix} 5 \\ 24 \end{bmatrix}$ as a linear combination of $\vec{v_1} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 3 \\ -9 \end{bmatrix}$.

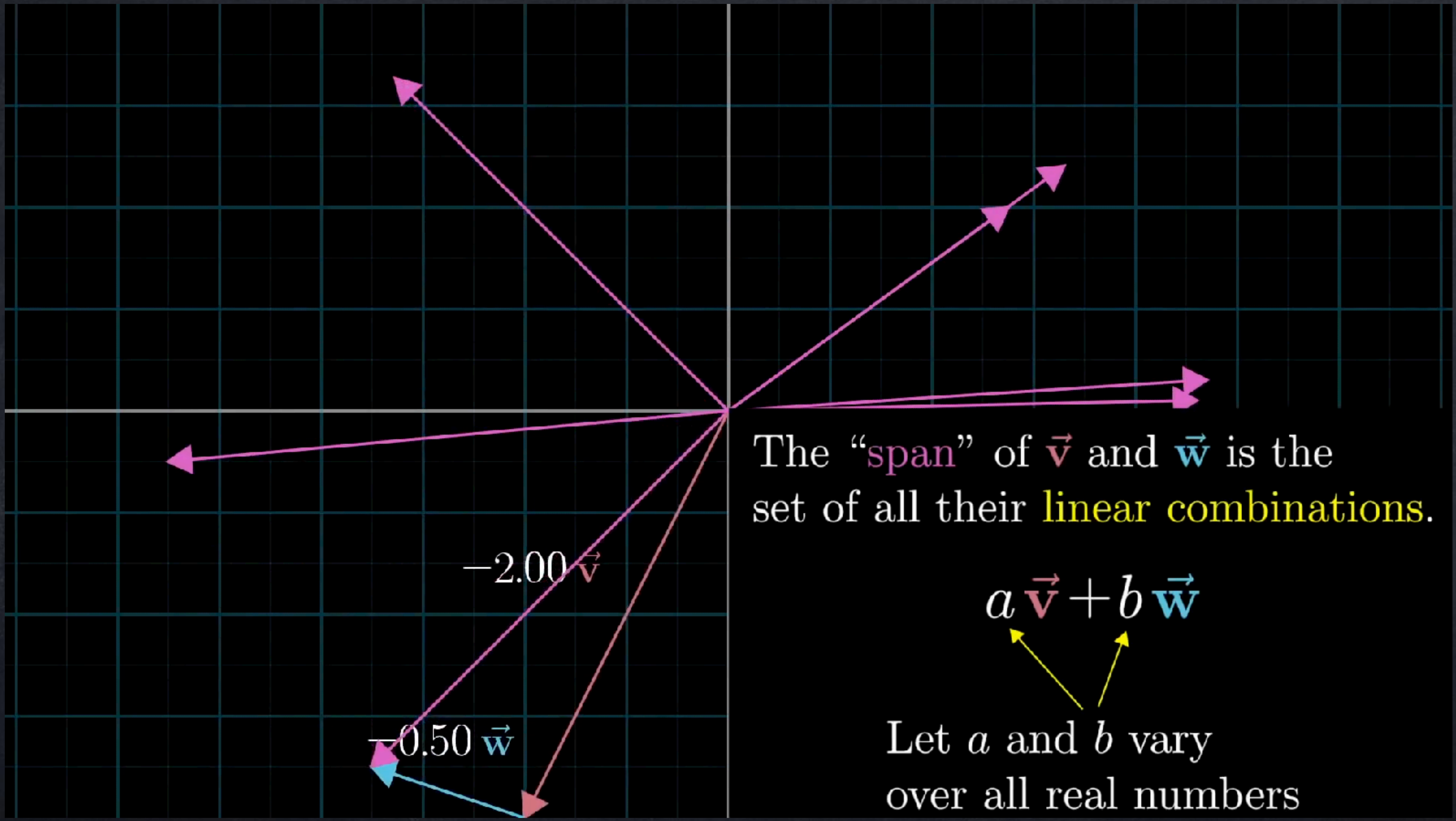
We want $x\begin{bmatrix} 5 \\ -2 \end{bmatrix} + y\begin{bmatrix} 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 5 \\ 24 \end{bmatrix}$, so we must solve the system $\begin{cases} 5x+3y=5 \\ -2x-9y=24 \end{cases}$.

Solution: $x = 3$, $y = -10/3$. Therefore $\begin{bmatrix} 5 \\ 24 \end{bmatrix} = 3\begin{bmatrix} 5 \\ -2 \end{bmatrix} + (-10/3)\begin{bmatrix} 3 \\ -9 \end{bmatrix}$.

Example 2: $\begin{bmatrix} 5 \\ 24 \end{bmatrix}$ *cannot* be written as a linear combination of $\overrightarrow{v_1} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $\overrightarrow{v_2} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$. Why?

Equations:

Pictures:



from 3Blue1Brown — youtu.be/k7RM-ot2NWY

In symbols, the span of $\vec{v}$ and $\vec{w}$ is the *set*

$$\{a\vec{v} + b\vec{w} \ : \ a, b \in \mathbb{R}\}.$$

This could be

- just the origin (here $\vec{v} = \vec{w} = \vec{0}$).
- a line through the origin (here $\vec{w} = s\vec{v}$ for some $s \in \mathbb{R}$).
- a plane. In 2D, the plane is "everything".
  In 3D, a plane is like an infinite flat sheet of paper.

Question: What does

$$\{\vec{v} + b\vec{w} \ : \ b \in \mathbb{R}\}$$

look like?

Parametric equations

Sometimes it's easiest to describe a shape using an extra variable in addition to x and y (and z in 3D).

Example 1

$$\begin{cases} x = 6\cos(t) \\ y = 6\sin(t) \end{cases}$$

Example 2

$$\begin{cases} x = 9^t \\ y = 3^t \end{cases}$$

Example 3

$$\begin{cases} x = 2 + t \\ y = 4 - t \end{cases}$$

Example 3 describes a straight line.

Lines

The line through point $\vec{p}$ parallel to the vector $\vec{d}$ can be described by the equation

$$\vec{r} = \vec{p} + t\,\vec{d}$$

the variable t is a parameter (sometimes s is used instead).

A vector parallel to a line is called a **direction vector** for that line.

This whole slide is good for 2D or 3D!

Lines

In 3D, the single vector equation $\vec{r} = \vec{p} + t\,\vec{d}$ is really the three equations:

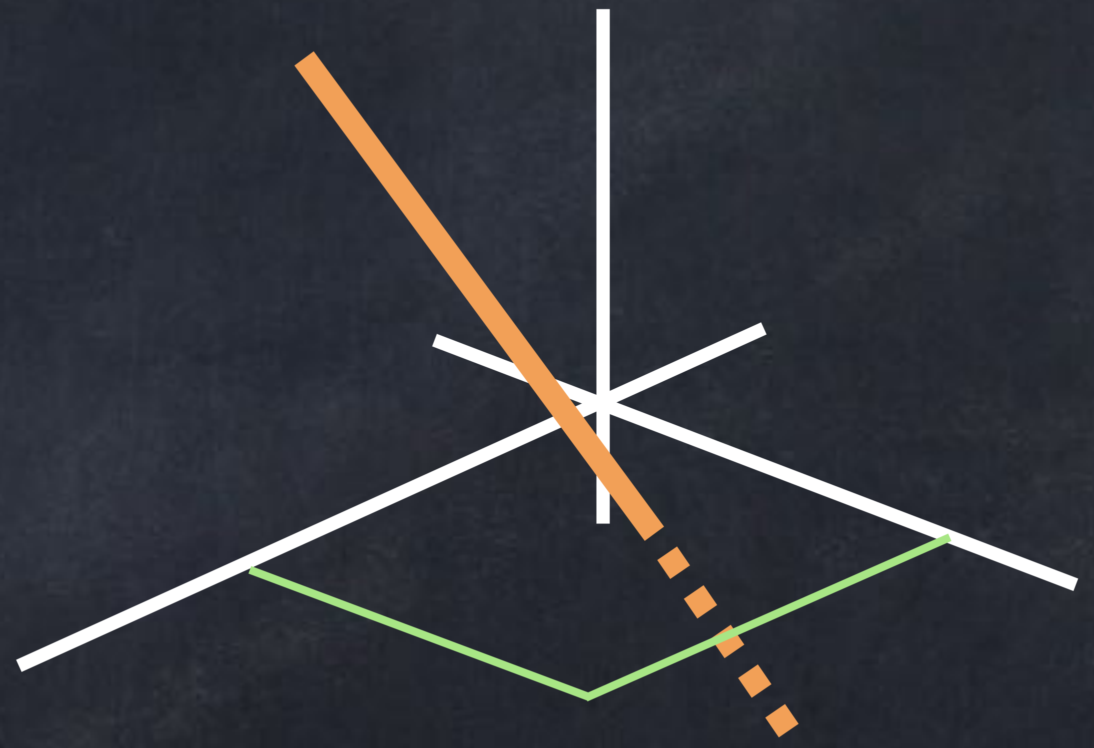
$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

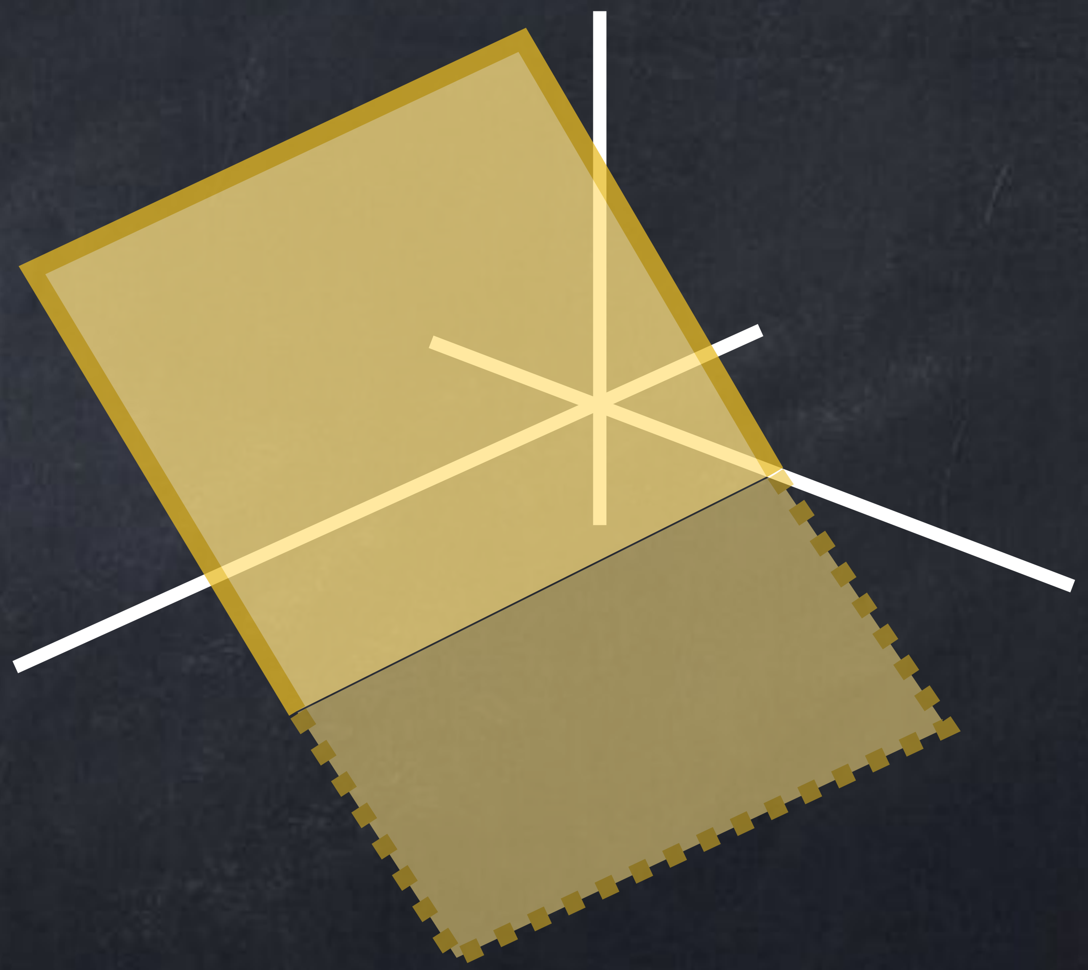
This is another common format for the for the line through (x_0, y_0, z_0) parallel to $\vec{d} = [a, b, c]$.

Lines and planes

You will need to be able to work *both* visually *and* with equations/symbols about

- lines in 2D
- lines in 3D
- planes in 3D





Task 1: Give an equation for the line L that goes through the point $(1,0,1)$ and is parallel to the vector $[5,1,6]$.

$$x = 1 + 5t, \; y = t, \; z = 1 + 6t$$

Creating an equation for a line is “easy” when you are given

- a point on the line and
- a direction vector.

It can be harder when you have to figure out one or both of those from other information.

Give an equation for the line L_1 that goes through the point $(6, 2, 1)$ and is parallel to the line

$$L_2: \quad x = 4 + 2t, \;\; y = -1 + 4t, \;\; z = 5 - t.$$

Direction vector of L_2 is [2,4,-1] (that's what's multiplied by t).

Using this as $\vec{d}$ for L_1 will make the lines parallel!

Line through (6,2,1) parallel to [2,4,-1] is

$$\boxed{x = 6 + 2t, \; y = 2 + 4t, \; z = 1 - t.}$$

Remember: Line through point $\vec{p}$ with direction vector $\vec{d}$ has equation $\vec{r} = \vec{p} + t\vec{d}$.

Two lines in 2D must be one of these:

- the same line,
- intersecting at exactly one point,
- parallel. ←

In 2D, the only way two lines can have no points in common is when the lines are parallel.

Two lines in 3D must be one of these:

- the same line,
- intersecting at exactly one point,
- parallel (definition: having parallel direction vectors),
- **skew** (definition: not fitting any of the previous three categories!).

- Are the lines

$$L_1: \quad x = 2 - t, \quad y = 1 + 2t, \quad z = 4 + t,$$
$$L_2: \quad x = -1 + s, \quad y = 7 - 3s, \quad z = 7 + s$$

intersecting, parallel, or skew?

- If they intersect, find the point where they intersect.

Intersect at (-1, 7, 7).
(This is t = 3 and s = 0.)

Warning: these could be written

$$L_1: \quad x = 2 - t, \quad y = 1 + 2t, \quad z = 4 + t,$$
$$L_2: \quad x = -1 + t, \quad y = 7 - 3t, \quad z = 7 + t$$

but the task would be the same.

Dot product

The **dot product** of two vectors $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$,

also called the **scalar product** or **inner product**, is written as $\vec{a} \cdot \vec{b}$ (said out loud as "A dot B"). It is a *number* that can be computed as either

- $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

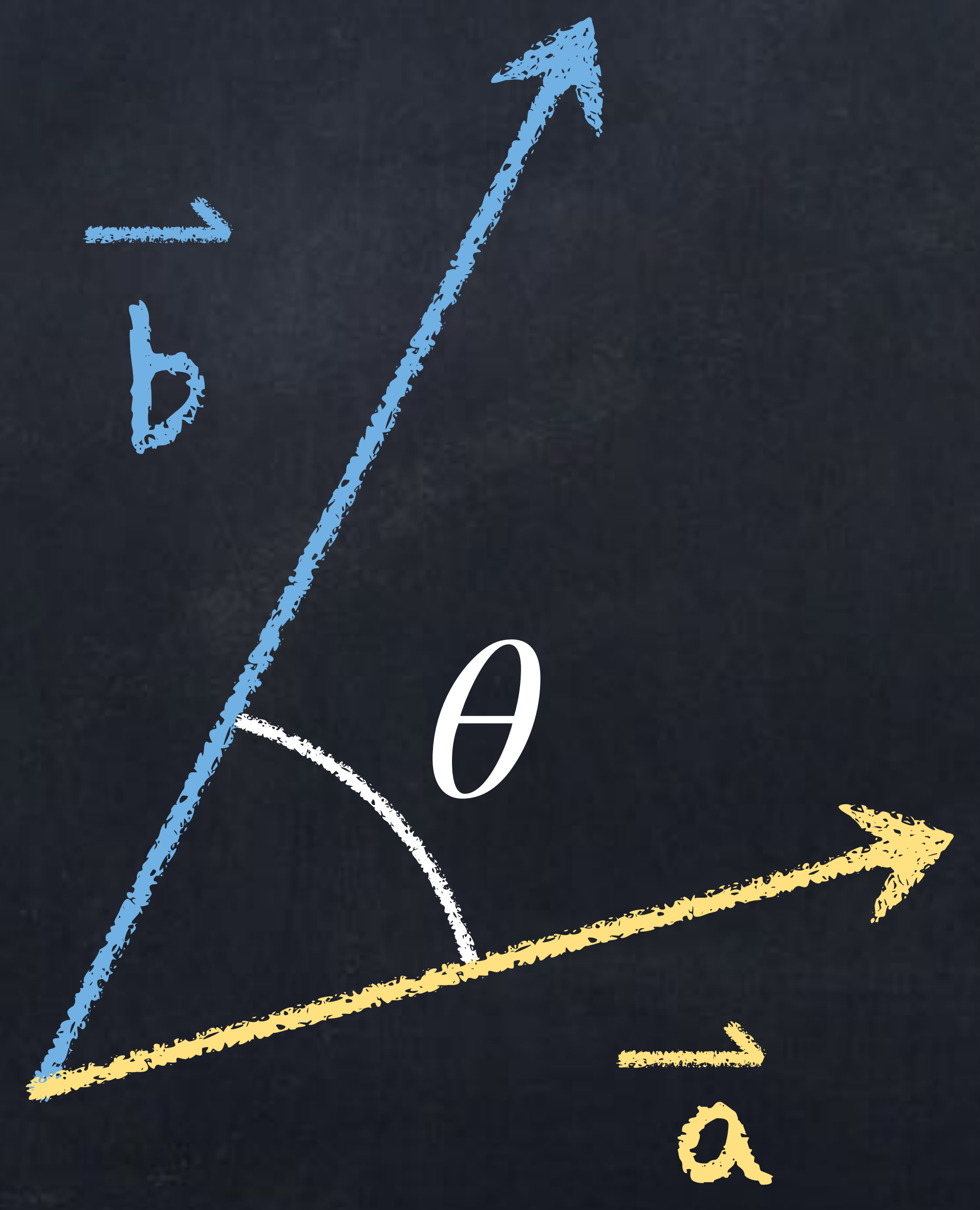
or

- $\vec{a} \cdot \vec{b} = \left|\vec{a}\right| \left|\vec{b}\right| \cos\left(\text{angle between } \vec{a} \text{ and } \vec{b}\right)$.

Using

- $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots$
- $\vec{a} \cdot \vec{b} = \left|\vec{a}\right| \left|\vec{b}\right| \cos\left(\text{angle between } \vec{a} \text{ and } \vec{b}\right)$

together, we can find the angle between vectors.



Example: Find the angle between $\langle \sqrt{3}, 1 \rangle$ and $\langle 0, 7 \rangle$.

$\vec{a}$ $\vec{b}$

$|\vec{a}| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3+1} = 2$

$|\vec{b}| = \sqrt{0^2 + 7^2} = 7$

So $\vec{a} \cdot \vec{b} = (2)(7)\cos\theta$.

But also $\vec{a} \cdot \vec{b} = (\sqrt{3})(0) + (1)(7) = 7$, so

$(2)(7)\cos\theta = 7 \rightarrow \cos\theta = 1/2 \rightarrow \boxed{\theta = 60^\circ}$