Math 1433

6 November 2023

Calculate the dot products

$$[3,6] \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} \text{ and } [8,0] \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \text{ and } [8,0] \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Dimensions

The **dimension** of a vector (list) is how many numbers are in the list.

- The dimension of the vector $\begin{bmatrix} -4 \\ 9 \end{bmatrix}$ is 2.
- The dimension of the vector $\begin{bmatrix} 57 \\ 0 \\ {}^1/_2 \end{bmatrix}$ is 3.

⭐ In order to add, subtract, or take dot products of vectors, they must have the same dimension.

$[2, 2] \cdot [1, -2, 4, 9]$ is nonsense.

$\vec{u} + \vec{w}$ is a vector.

Assume $\vec{u}, \vec{v}, \vec{w}$ all have dimension 3.

$5 + \vec{v}$ is nonsense.

$5\,\vec{w}$ is a vector.

$\vec{u} \cdot \vec{w}$ is a scalar (a number).

$\vec{u} \times \vec{w}$ is a vector.

$\dfrac{\vec{u}}{\vec{w}}$ is nonsense.

If we think of $\frac{a}{b}$ as the answer to $b \times (?) = a$, we have a problem using vectors:

$$[2, 1, -2] \times [-2, 4, 3] = [11, -2, 10]$$
$$[2, 1, -2] \times [0, 5, 1] = [11, -2, 10]$$
$$[2, 1, -2] \times [2, 6, -1] = [11, -2, 10]$$

There is no such thing as division for cross product.

$\dfrac{\vec{u}}{12}$ is a vector.

Transformations of vectors

Pre-viously

The functions you study in school and in Analysis 1 are usually from $\mathbb{R}$ to $\mathbb{R}$, meaning the input and output are numbers.

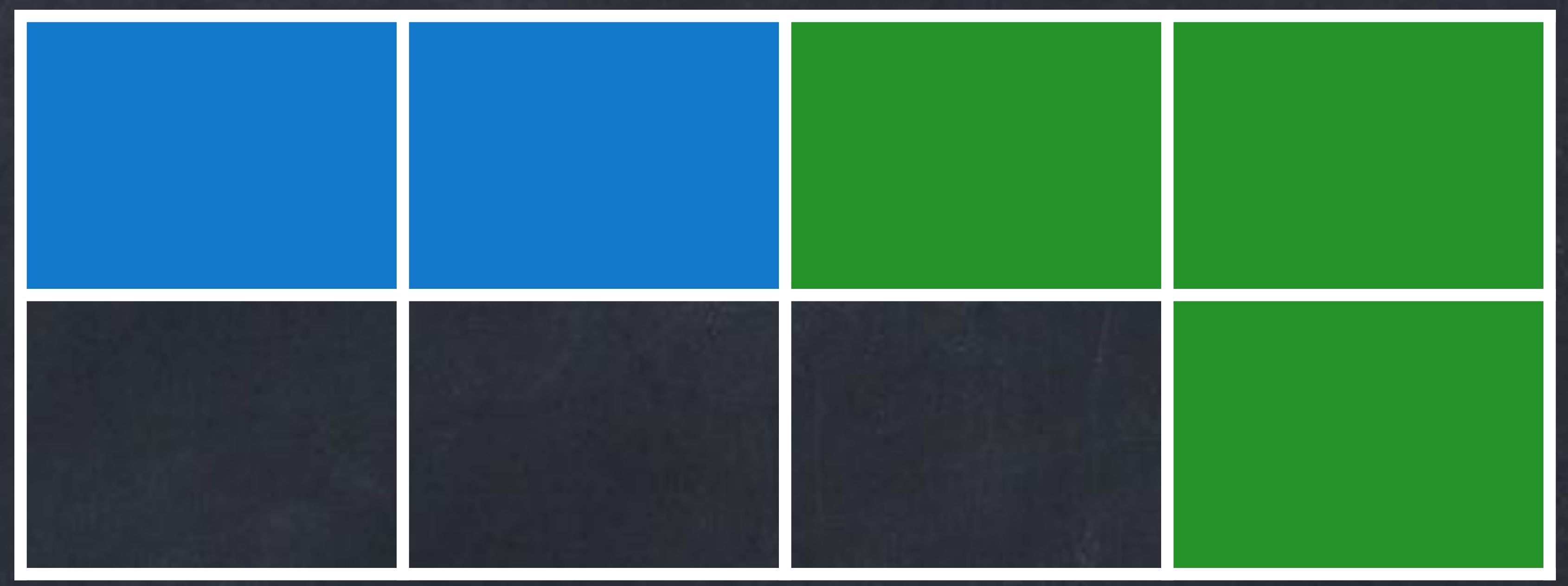
An example of a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ could be $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ e^x \end{bmatrix}$.

We can also write that as

$$f(x\hat{\imath} + y\hat{\jmath}) = (x-y)\hat{\imath} + e^x\hat{\jmath} \quad \text{or} \quad f(x,y) = \left(x-y, e^x\right).$$

Often, the word **transformation** is used instead of function when talking about vectors.

Learning the formula $\frac{a}{b} + \frac{c}{d} = \frac{bc + ad}{bd}$ without ever seeing pictures like



seems ridiculous. But that is what we will do for matrices today.

- The *reason* that matrix multiplication is calculated the way it is involves linear transformations.
- To actually do matrix calculations, it's easier to memorize a formula / rule.

QUIZZES

Quiz 3 next week:

- Lines
- Planes
- any previous material.

See Lists 1-2. I will provide $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$ for you on the quiz.

Quiz 4 the week after (20.10, a single-meeting day):

- Matrix multiplication *calculations*
- When is it possible to do $\vec{a} + \vec{b}$, $\vec{a} \times \vec{b}$, $A + B$, AB, etc.

See List 3.

Vector

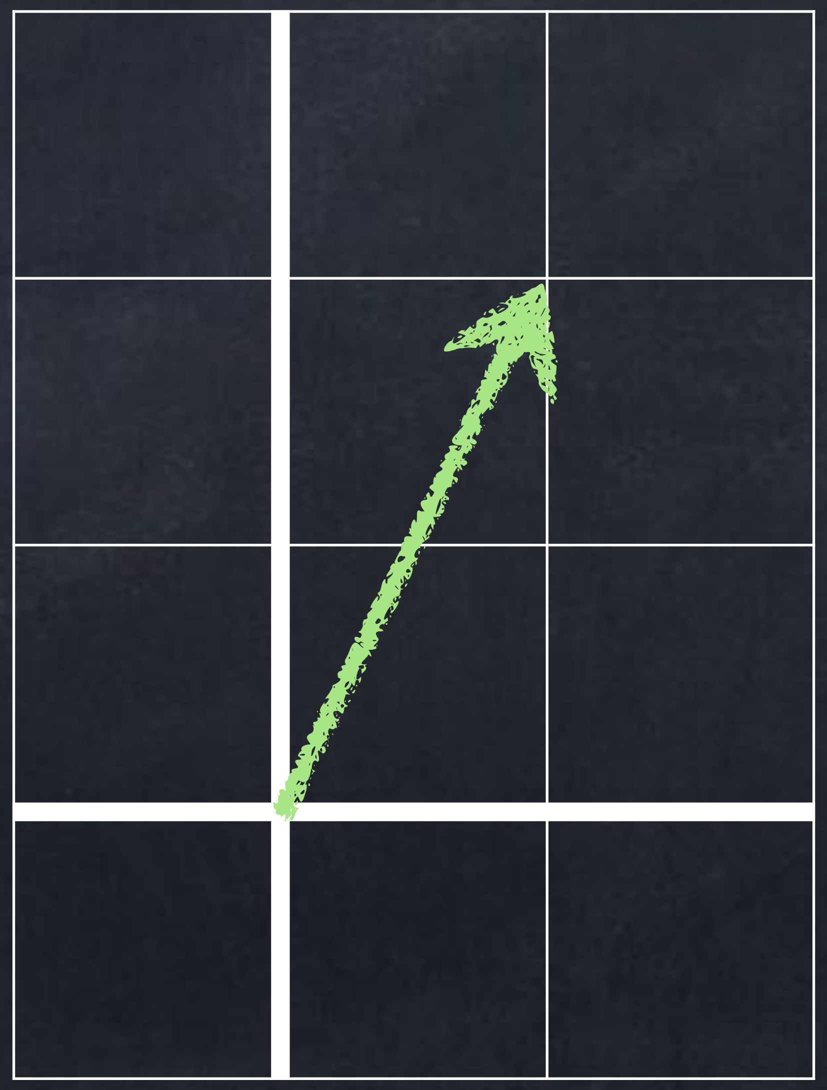
A vector is…

- …a list of numbers.
- …an arrow.
- …a point.

$$[1, 2]$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\hat{\imath} + 2\hat{\jmath}$$



Matrix

A **matrix** is…

- …a rectangle of numbers.
- …a list of vectors, where each vector is a row.
- …a list of vectors, where each vector is a column.
- …(other ways to think about matrices will come later).

$$\begin{bmatrix} 5 & 8 \\ -2 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 4 & -4 \\ 2 & 12 \end{bmatrix} \qquad \begin{bmatrix} -8 & 19 & 4 & 4 \\ 19 & -4 & 2 & 6 \\ 15 & 8 & 2 & 16 \\ 3 & 14 & 0 & 12 \end{bmatrix}$$

Matrix

One **matrix** (“may-tricks” [meɪtrɪks]), two **matrices** (“may-trih-sees” [meɪtrɪsiz]).

We usually use a capital letter (no $\vec{}$ or other mark) for a matrix variable.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 3 & -5 & -10 \end{bmatrix}, \qquad M = \begin{bmatrix} 5 & 0 \\ 0 & \frac{1}{5} \end{bmatrix}.$$

The entries in a matrix are sometimes given two subscripts:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix} \text{ or } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

Row, column, dimensions

In the matrix $\begin{bmatrix} -1 & 2 & 21 \\ 13 & -1 & -7 \\ 10 & -9 & 13 \end{bmatrix}$,

- the **rows** are $\begin{bmatrix} -1 & 2 & 21 \end{bmatrix}$ and $\begin{bmatrix} 13 & -1 & -7 \end{bmatrix}$ and $\begin{bmatrix} 10 & -9 & 13 \end{bmatrix}$.
- the **columns** are $\begin{bmatrix} -1 \\ 13 \\ 10 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ -9 \end{bmatrix}$ and $\begin{bmatrix} 21 \\ -7 \\ 13 \end{bmatrix}$.
- the **main diagonal** is $\begin{matrix} -1 & & \\ & -1 & \\ & & 13 \end{matrix}$.

Row, column, dimensions

In the matrix $\begin{bmatrix} -1 & 2 & 21 \\ 13 & -1 & -7 \\ 10 & -9 & 13 \end{bmatrix}$,

- the **rows** are $[-1 \quad 2 \quad 21]$ and $[13 \quad -1 \quad -7]$ and $[10 \quad -9 \quad 13]$.
- the **columns** are $\begin{bmatrix} -1 \\ 13 \\ 10 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ -9 \end{bmatrix}$ and $\begin{bmatrix} 21 \\ -7 \\ 13 \end{bmatrix}$.
- the **main diagonal** is $\begin{matrix} -1 & & \\ & -1 & \\ & & 13 \end{matrix}$.

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- the **rows** are $[-1 \quad 2 \quad 21]$ and $[13 \quad -1 \quad -7]$ and $[10 \quad -9 \quad 13]$.
- the **columns** are $\begin{bmatrix} -1 \\ 13 \\ 10 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ -9 \end{bmatrix}$ and $\begin{bmatrix} 21 \\ -7 \\ 13 \end{bmatrix}$.
- the **main diagonal** is $\begin{matrix} -1 & & \\ & -1 & \\ & & 13 \end{matrix}$.

Dimensions

The dimension of the vector $\begin{bmatrix} -4 \\ 9 \end{bmatrix}$ is 2. (or 2x1 if we think of this as a matrix)

The dimension of the vector $\begin{bmatrix} 57 \\ 0 \\ {}^1/_2 \end{bmatrix}$ is 3. (or 3x1 if we think of this as a matrix)

The **dimensions** of the matrix $\begin{bmatrix} 8 & 5 & -1 \\ 0 & 4 & 4 \end{bmatrix}$ are 2×3 (aloud: "2 by 3").

We have to list both numbers! Dimensions 2×3 does not mean 6.

Dimensions

Always rows first, then columns.

For example a 3×2 matrix looks like $\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$, while a 2×3 matrix looks like $\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$. These are different dimensions.

A **square matrix** is one with the same number of rows as columns.

Matrix calculations

- scalar multiplication $9\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} = ?$
- addition/subtraction $\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} + \begin{bmatrix}5 & 6\\ 7 & 8\end{bmatrix} = ?$
- matrix times a vector $\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}10\\ 11\end{bmatrix} = ?$ (note: no · or × required)
- matrix multiplication $\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}5 & 6\\ 7 & 8\end{bmatrix} = ?$ (note: no · or × required)

Matrix calculations

- scalar multiplication $$9\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} = \begin{bmatrix}9 & 18\\ 27 & 36\end{bmatrix}$$
- addition/subtraction $$\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} + \begin{bmatrix}5 & 6\\ 7 & 8\end{bmatrix} = \begin{bmatrix}6 & 8\\ 10 & 12\end{bmatrix}$$

These two are exactly what you would expect 🙂

Official formulas: If $M = sA$ then $m_{ij} = s\,a_{ij}$, where s is a number.
If $M = A \pm B$ then $m_{ij} = a_{ij} \pm b_{ij}$.

Two subtraction examples:

$$\begin{bmatrix} 5 & 1 \\ 2 & 9 \\ 0 & 7 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -2 & 5 \\ 5 & 1 \end{bmatrix} = ?$$

(easy)

$$\begin{bmatrix} 5 & 1 \\ 2 & 9 \\ 0 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 8 \\ -1 & 5 & 6 \end{bmatrix} = ?$$

This DOES NOT EXIST.

🌟 The sum $(A + B)$ and difference $(A - B)$ of two matrices only exists if the matrices have exactly the same dimensions.

Matrix multiplication

Example 1: How can we calculate $\begin{bmatrix} 3 & 6 \\ 8 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$? There are two ways.

Using linear combinations: $\begin{bmatrix} 3 & 6 \\ 8 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 1 \begin{bmatrix} 3 \\ 8 \end{bmatrix} + (-2) \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} -9 \\ 8 \end{bmatrix}$

Using dot products: $\begin{bmatrix} 3 & 6 \\ 8 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} (3,6) \cdot (1,-2) \\ (8,0) \cdot (1,-2) \end{bmatrix} = \begin{bmatrix} -9 \\ 8 \end{bmatrix}$

$[3,6] \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} = -9$

and

$[8,0] \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 8$

Rule: if $\vec{w} = A\vec{v}$ then w_i = (row i of A) $\cdot \vec{v}$.

Example 2: $\begin{bmatrix} 3 & 6 \\ 8 & 0 \\ 2 & 2 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 8 \\ -2 \\ 15 \end{bmatrix}$

Rule: if $\overrightarrow{w} = A\vec{v}$ then w_i = (row i of A) $\cdot \vec{v}$.

Example 3: $\begin{bmatrix} 3 & 6 \\ 8 & 0 \\ 2 & 2 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 4 \\ 9 \end{bmatrix}$ DOES NOT EXIST.

from before

⭐ In order to add, subtract, or take dot products of vectors, they must have the same dimension.

Rule: if $\vec{w} = A\vec{v}$ then w_i = (row i of A) $\cdot$ $\vec{v}$.

Rule: if $M = AB$ then m_{ij} = (row i of A) $\cdot$ (column j of B).

Example 4: $\begin{bmatrix} 4 & 1 & 0 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 9 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 15 & 16 \end{bmatrix}$

$$\begin{array}{ccc|cc} & & & 1 & -1 \\ & & & 2 & 9 \\ & & & 3 & 1 \\ \hline 4 & 1 & 0 & 6 & 5 \\ -2 & 1 & 5 & 15 & 16 \end{array}$$

$[4,1,0] \cdot [1,2,3] = 4+2+0 = 6$

$[4,1,0] \cdot [-1,9,1] = -4+9+0 = 5$

Rule: if $\vec{w} = A\vec{v}$ then w_i = (row i of A) $\cdot$ $\vec{v}$.

Rule: if $M = AB$ then m_{ij} = (row i of A) $\cdot$ (column j of B).

Example 4: $\begin{bmatrix} 4 & 1 & 0 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 9 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 15 & 16 \end{bmatrix}$

$$\begin{array}{ccc|cc} & & & 1 & -1 \\ & & & 2 & 9 \\ & & & 3 & 1 \\ \hline 4 & 1 & 0 & 6 & 5 \\ -2 & 1 & 5 & 15 & 16 \end{array}$$

$[-2,1,5] \cdot [1,2,3] = -2+2+15 = 15$

$[-2,1,5] \cdot [-1,9,1] = 2+9+5 = 16$

Rule: if $\vec{w} = A\vec{v}$ then w_i = (row i of A) $\cdot \vec{v}$.

Rule: if $M = AB$ then m_{ij} = (row i of A) $\cdot$ (column j of B).

Example 4: $\begin{bmatrix} 4 & 1 & 0 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 9 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 15 & 16 \end{bmatrix}$

Alternatively:

$$(1)\begin{bmatrix} 4 \\ -2 \end{bmatrix} + (2)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (3)\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 15 \end{bmatrix}$$

and

$$(-1)\begin{bmatrix} 4 \\ -2 \end{bmatrix} + (9)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 16 \end{bmatrix}$$

Rule: if $M = AB$ then m_{ij} = (row i of A) · (column j of B).

Example 5: $\begin{bmatrix} 4 & 1 & 0 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 6 & -3 \end{bmatrix}$ impossible!

because [4,1,0] · [4,6] is impossible

2 x 3 2 x 2

The "inner" numbers must agree for AB to exist.

🌟 The matrix multiplication AB is only possible if (# of columns of A) = (# of rows of B).

Rule: if $M = AB$ then m_{ij} = (row i of A) · (column j of B).

Example 4: again

$$\begin{bmatrix} 4 & 1 & 0 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 9 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 15 & 16 \end{bmatrix}$$

2 x 3   3 x 2   2 x 2

The "inner" numbers must agree for AB to exist.

The "outer" numbers give the dimensions of AB.