Math 1433

20 November 2023

Quiz/exam schedule

(It's on the course website calendar.)

- Today: Quiz 4 – matrix calculations
- 11 December: Midterm exam 🎉
- January: Quiz 5 and 6
- February: Final exam 🎉 (and optional retake).

Identity

The **$n \times n$ identity matrix**, written $I_{n\times n}$ or I_n or just I, is a special matrix such that, if the products exist,

- $I\vec{v} = \vec{v}$ for any vector $\vec{v}$,
- $IM = M$ for any matrix M,
- $MI = M$ for any matrix M.

In a way, each matrix $I_{n\times n}$ acts like the number 1 because multiplying by it does not cause any change.

Formulas: $I_{2\times2} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. $I_{3\times3} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$. $I_{4\times4} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$.

Inverse

Last time

The **inverse** of M, written M^{-1}, is the matrix for which $MM^{-1} = M^{-1}M = I$.

For a 2×2 matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have

$$M^{-1} = \frac{1}{\det(M)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

where

$$\det(M) = ad - bc$$

is called the **determinant** of M.

A square matrix has an inverse if and only if its determinant is *not* zero.

For larger matrices the formulas are worse, but the boxed fact is still true.

Arithmetic

Algebra

Geometry

Analysis

Arithmetic

Task: Divide 30 by 12.

Answer:

$$\frac{5}{2}$$

or

$$2+\frac{1}{2}$$

or

$$2.5$$

or

$$2 \text{ remainder } 6$$

Algebra

Task: Solve $12x = 30$.

Answer:

$$x = \frac{5}{2}$$

or…

You need to be comfortable with calculations before solving equations.

Arithmetic

Task: Multiply $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

Algebra

Task: Solve $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} X = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

matrix

You need to be comfortable with ^ calculations before solving equations.

Algebra

Task: Solve $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} X = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

This has the format

$$AX = B,$$

so we can solve it by

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

Note: we cannot say

$$X = BA^{-1}$$

because $A^{-1}B$ is not the same as $A^{-1}B$.

$$X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ 4 \end{bmatrix}$$

You need to be comfortable with matrix calculations before solving equations.

Determinant

For 2×2 matrices, calculating the determinant is easy:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

For 3×3 matrices, calculating the determinant is more work:

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b\det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c\det\begin{pmatrix} d & e \\ g & h \end{pmatrix}.$$

There is a nice pattern to help you remember/use this formula…

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - \cdots$$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \cdots - b\det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + \cdots$$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \cdots + c\det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

Calculate $\det\begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix}$.

Calculate $\det\begin{pmatrix} 3 & -2 \\ 3 & 4 \end{pmatrix}$.

Calculate $\det\begin{pmatrix} 5 & 3 \\ p & 4 \end{pmatrix}$. $= 20 - 3p$

Calculate $\det\begin{pmatrix} 5 & -1 & 0 \\ 3 & 1 & -2 \\ 3 & 0 & 4 \end{pmatrix}$.

easy to forget

$$= 5\det\begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix} - (-1)\det\begin{pmatrix} 3 & -2 \\ 3 & 4 \end{pmatrix} + 0\det\begin{pmatrix} 3 & 1 \\ 3 & 0 \end{pmatrix}$$

$$= 5(4) - (-1)(18) + 0(-3)$$

$$= 20 + 18 + 0$$

$$= 38$$

Calculate $\det\begin{pmatrix} i & j & k \\ 3 & 1 & -2 \\ 3 & 0 & 4 \end{pmatrix}$.

$$= i \det\begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix} - j \det\begin{pmatrix} 3 & -2 \\ 3 & 4 \end{pmatrix} + k \det\begin{pmatrix} 3 & 1 \\ 3 & 0 \end{pmatrix}$$

$$= 4i - 18j - 3k$$

Magic: vector cross product is exactly this!

$$[3, 1, -2] \times [3, 0, 4] = 4\hat{\imath} - 18\hat{\jmath} - 3\hat{k}.$$