Math 1433

27 November 2023

Warm-ups:

Solve $\begin{cases} 3x - 6y + z = 7 \\ x + 8y - z = 4 \\ -x + y + 2z = 3 \end{cases}$ and then $\begin{cases} 3x - 6y + z = 7 \\ x + 8y - z = 4 \\ 2x + 16y - 2z = 1 \end{cases}$

$(x,y,z) = \left(\frac{5}{2}, \frac{1}{2}, \frac{5}{2}\right)$

No solution!

Quiz/exam schedule

(It's on the course website calendar.)

- 11 December: Midterm exam 🎉
- January: Quiz 5 and 6
- February: Final exam 🎉 (and optional retake).

Applications of matrices

Matrices (the plural of “matrix”) can be used for

- *systems of equations*
- geometry / linear transformations
- network/graph analysis
- probability and statistics
- cryptography
- image compression
- physics - optics, electronics, quantum

and more.

Systems of linear equations

A **linear equation** is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $x_1, ..., x_n$ are variables and $a_1, ..., a_n, b$ are coefficients (usually each a_i is a just a constant number, but it could be some expression that does not involve any x_i).

A **system of linear equations** (or just **system**) is a collection of linear equations with the same variables.

- Some equations may have coefficients of 0 for some variables, so we might not see every variable appear in every equation.
- Often, we will have the same number of variables as equations, but this is not required.

Systems of linear equations

Examples:

$$\begin{cases} 3x - 7y = 4 \\ x + 8y = 2 \end{cases}$$

$$\begin{cases} 7x + 2y + 9z = -9 \\ 7x + 2y + 9z = 4 \\ -4x - 3y + 5z = 2 \end{cases}$$

$$\begin{cases} 7a + 2b + 9c = -9 \\ 8a + 6c = 0 \\ -4a - 3b + 5c = 2 \end{cases}$$

$$\begin{cases} 3s + 2 = 7t - 1 \\ s + 8 = 2t + 12 \\ 4s = -t + 5 \end{cases}$$

$$\begin{cases} 7x + 2y + 9z = -9 \\ -4x - 3y + kz = 2 \end{cases}$$

$$\begin{cases} 6x_1 + 2x_2 - 5x_3 + x_4 = 1 \\ 5x_1 - 7x_3 + 2x_4 = 3 \end{cases}$$

Systems of equations

Finding values or formulas for the variables in a system is called "solving" the system. Any assignment that makes all equations true is a **solution**.

Example: The only solution to

$$\begin{cases} 6a - b = 15 \\ 2a + b = 1 \end{cases}$$

is $(a, b) = (3, -2)$.

Example: $\begin{cases} a^2 + b = 3 \\ a^2 - b = 7 \end{cases}$ has two solutions:

$$(a, b) = (\sqrt{5}, 2) \text{ and } (a, b) = (-\sqrt{5}, 2).$$

not linear

Number of eqns and variables

A system of equations is called **consistent** if at least one solution exists.
It is called **inconsistent** if no solutions exist.

An **overdetermined** system has more equations than variables.

- Overdetermined systems are *usually* (but not always) inconsistent.

An **underdetermined** system has fewer equations than variables.

- Underdetermined systems are *usually* (but not always) consistent.

Number of eqns and variables

Overdetermined

$$\begin{cases} 3x - 4y + 2z = 8 \\ x + 7y = 2 \\ 6x + y - 3z = 1 \\ 8y + z = 0 \\ 4x + 2z = 3 \end{cases}$$

$$\begin{cases} 4x - 3y = 1 \\ x + 5y = 6 \\ 2x + y = 3 \end{cases}$$

Underdetermined

$$\begin{cases} 3x - 4y + 2z = 8 \\ 3x - 4y + 2z = 1 \end{cases}$$

$$\begin{cases} 3x - 4y + 2z = 8 \\ x + y = 2 \end{cases}$$

There are *many* methods to solve systems of linear equations by hand. Some of the most common are

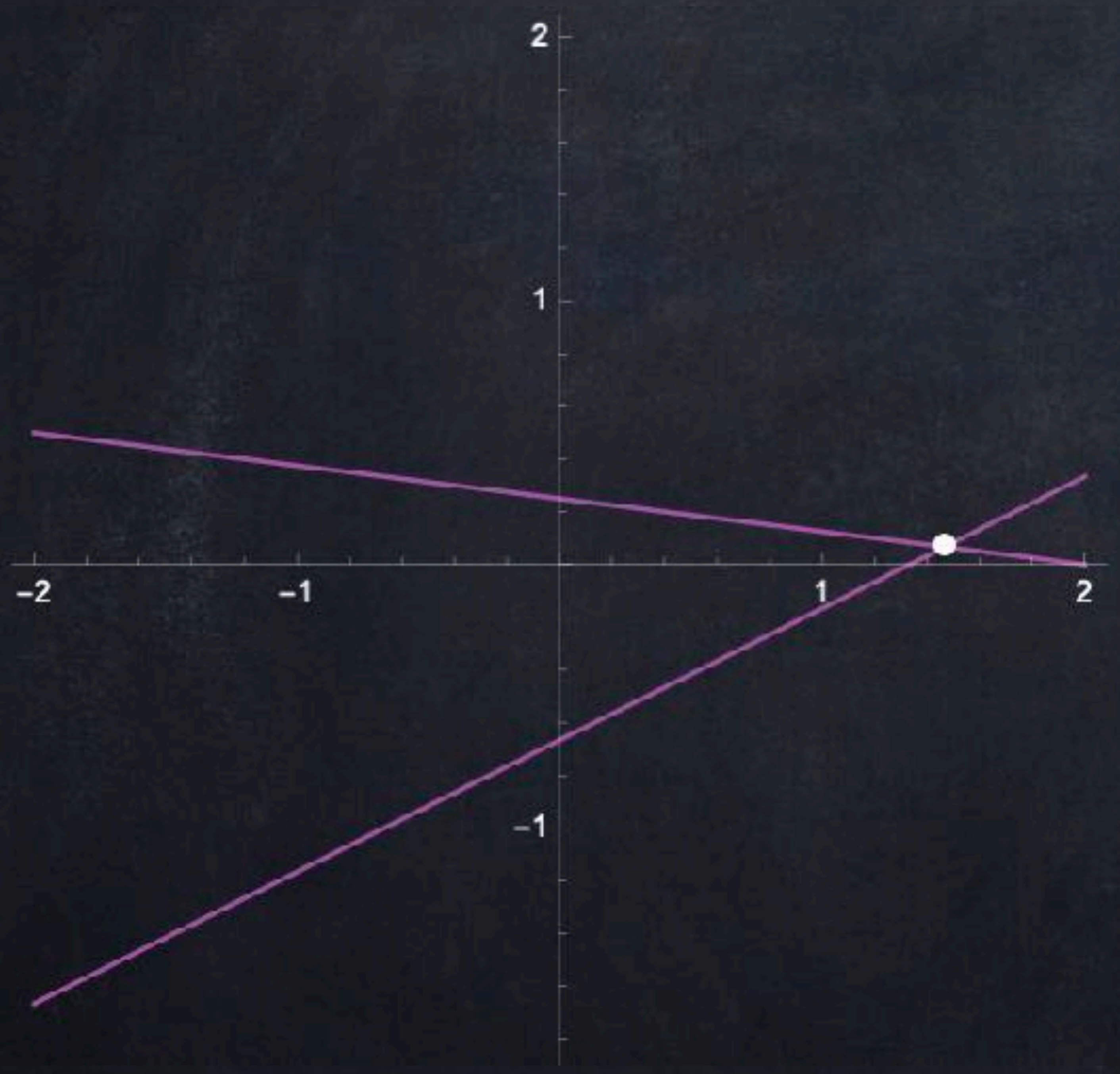
- Substitution
- Elimination
- Matrix inverse
- Cramer’s Rule.

Of course, computers can solve systems of equations for us.

Question: How many solutions can a *linear* system have?

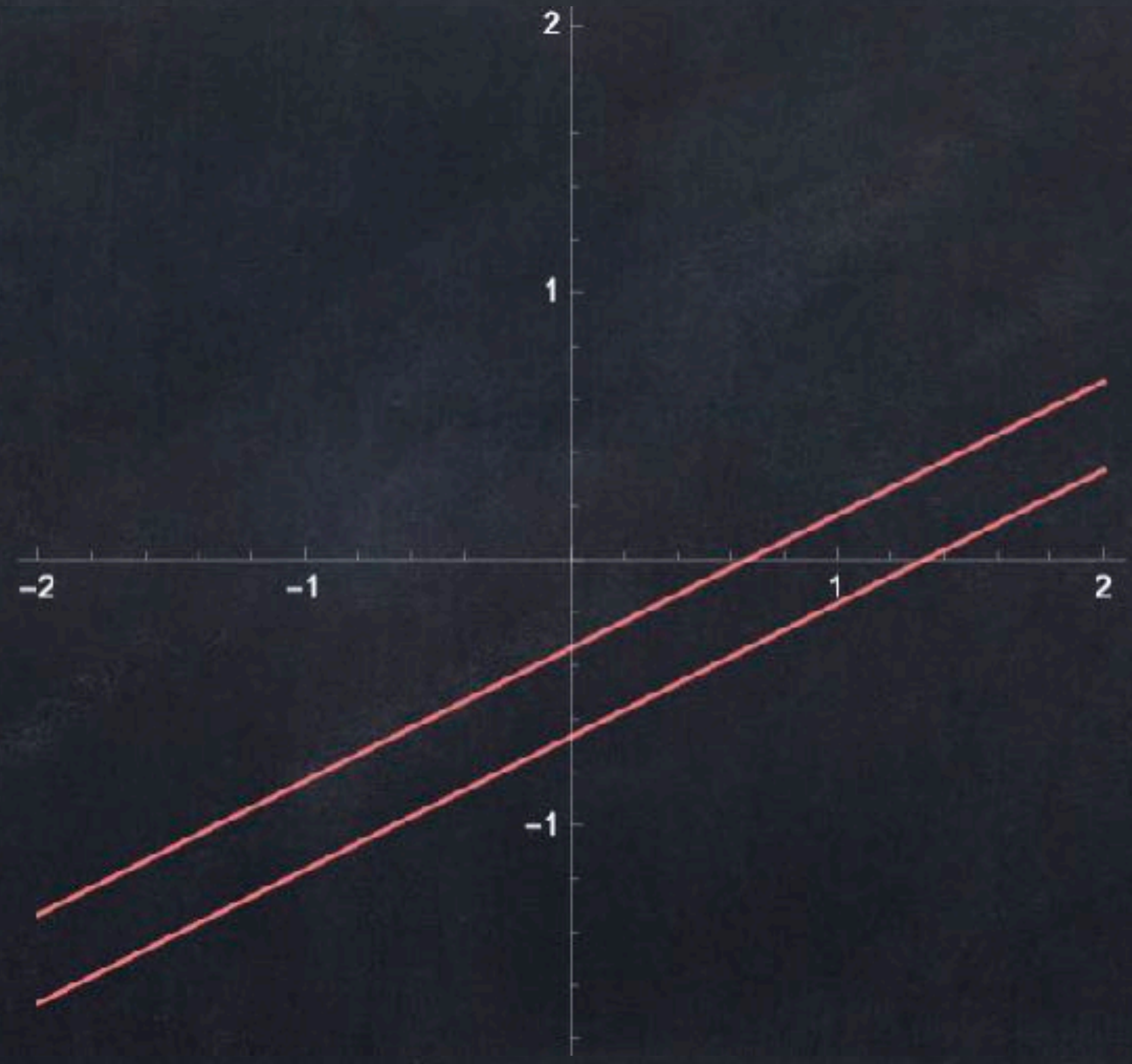
2 equations and 2 variables

$$\begin{cases} 3x - 6y = 4 \\ x + 8y = 2 \end{cases}$$



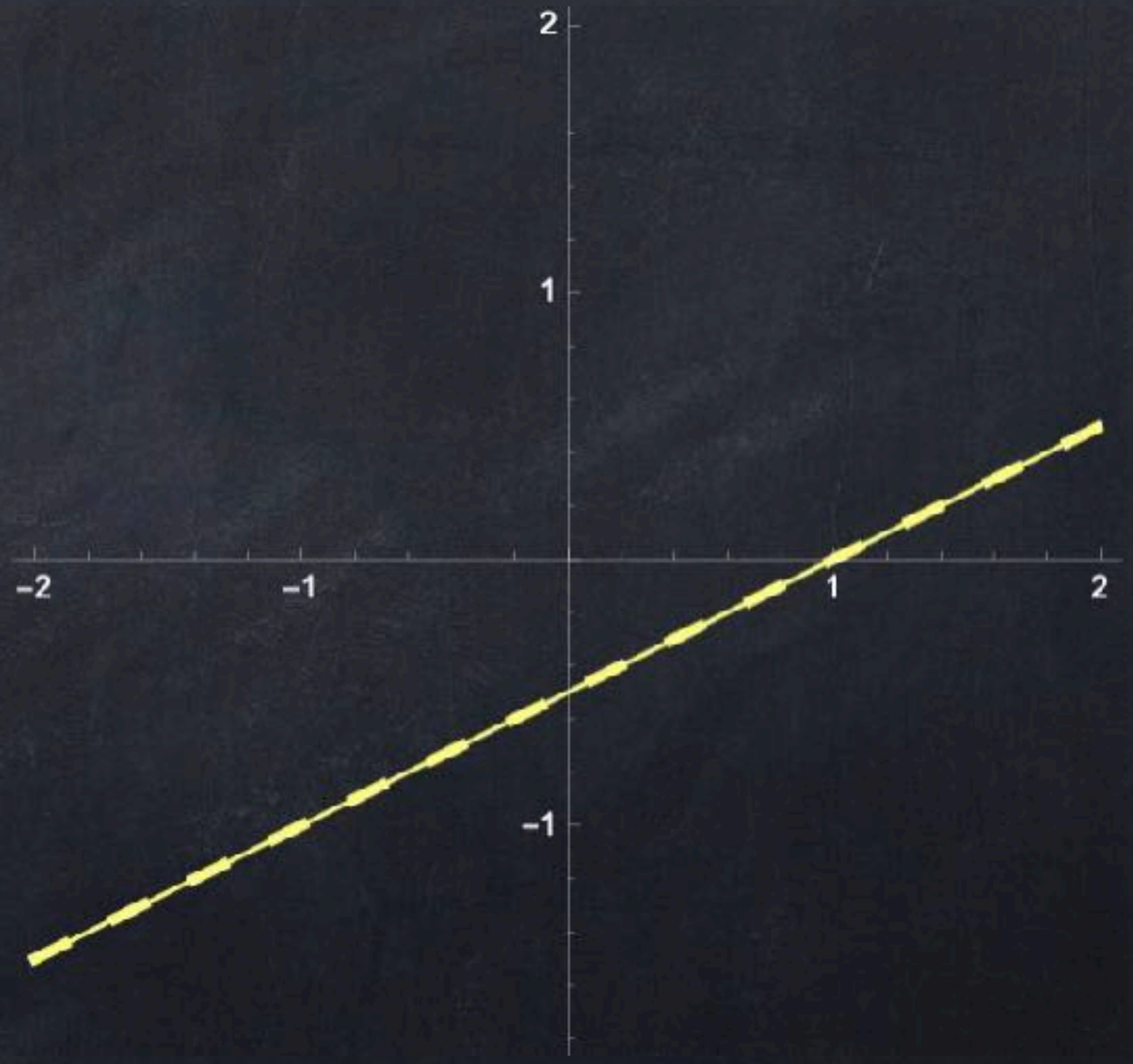
one solution

$$\begin{cases} 3x - 6y = 4 \\ 3x - 6y = 2 \end{cases}$$



no solutions

$$\begin{cases} 3x - 6y = 3 \\ x - 2y = 1 \end{cases}$$



infinitely many solutions

3 equations and 3 variables

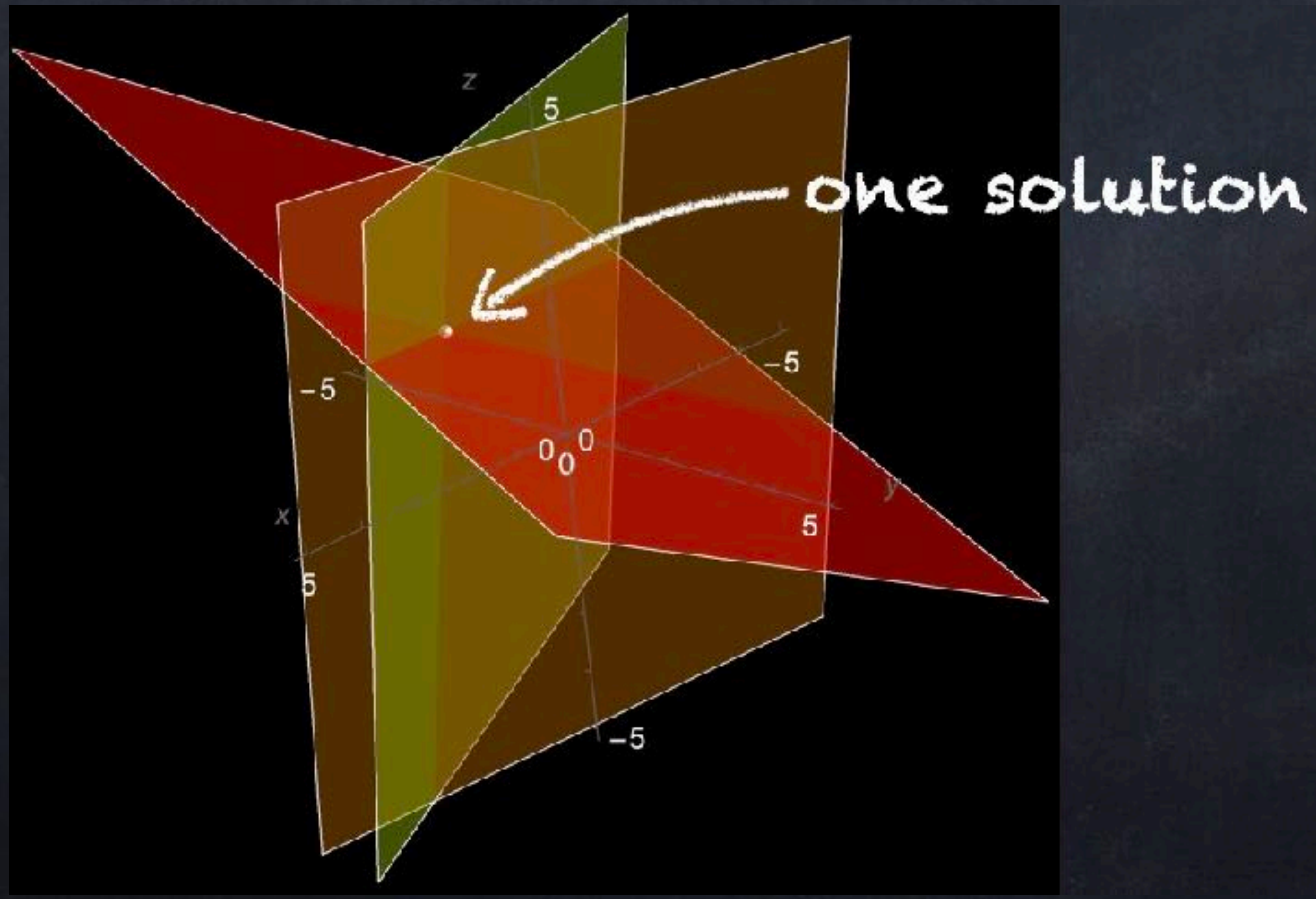
$$\begin{cases} 3x - 6y + z = 7 \\ x + 8y - z = 4 \\ -x + y + 2z = 3 \end{cases}$$

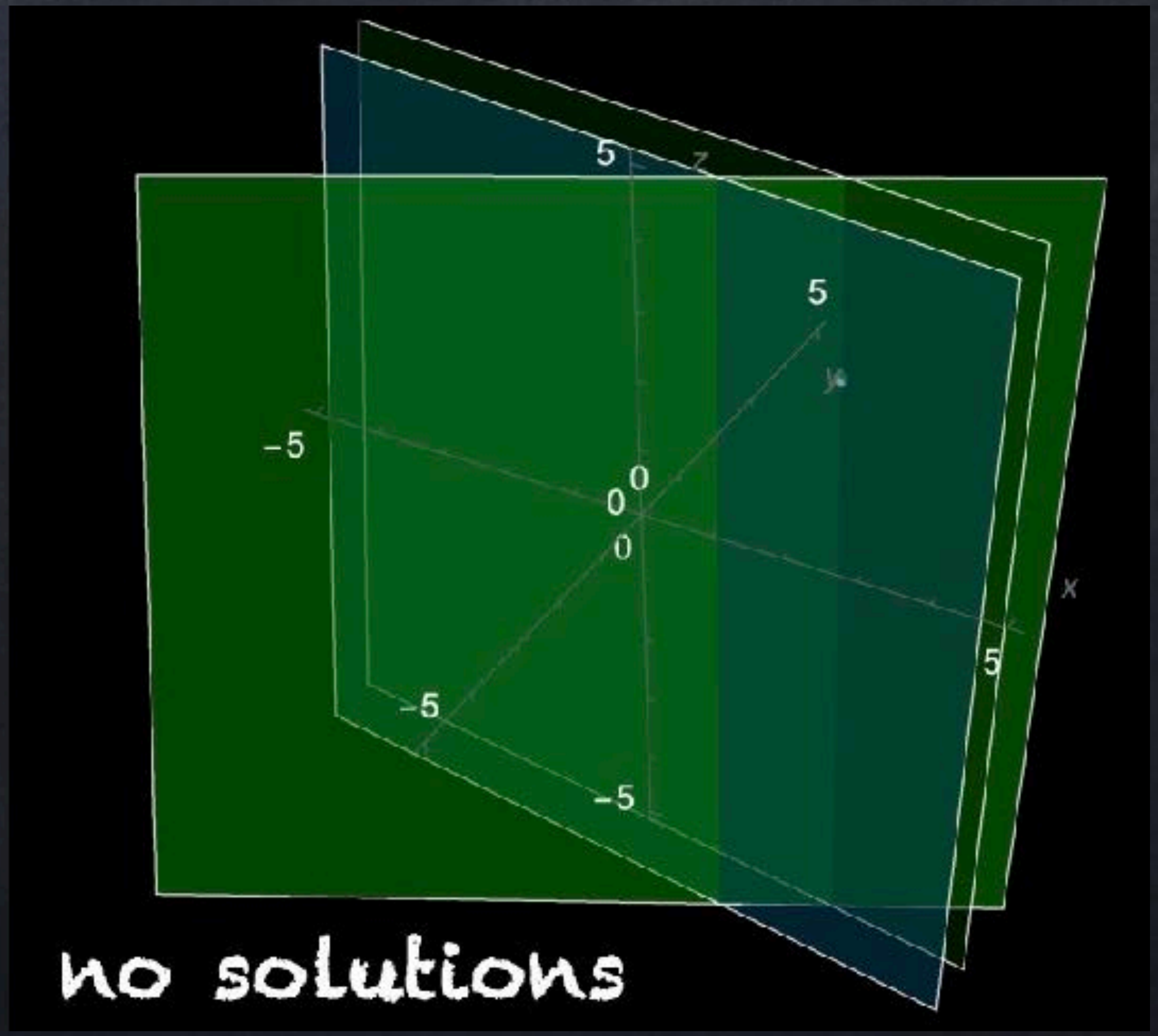
$$\begin{cases} 3x - 6y + z = 7 \\ x + 8y - z = 4 \\ 2x + 16y - 2z = 1 \end{cases}$$

3 equations and 3 variables

$$\begin{cases} 3x - 6y + z = 7 \\ x + 8y - z = 4 \\ -x + y + 2z = 3 \end{cases}$$

$$\begin{cases} 3x - 6y + z = 7 \\ x + 8y - z = 4 \\ 2x + 16y - 2z = 1 \end{cases}$$





We can draw other arrangements with multiple planes in 3D space.

Any linear system—with *any* number of variables and *any* number of equations—will have either

- **0** solutions,
- exactly **1** solution, or
- **infinitely many** solutions.

Non-linear equations can be very different (for example, $\begin{cases} x^2 + y^2 = 16 \\ x + y = 0 \end{cases}$ has exactly 2 solutions, but this can never happen for linear systems).

There are many methods to solve systems of linear equations by hand.

- Substitution
- Elimination
- Matrix inverse*
- Cramer's Rule*

Fewer calculations, but you have to be clever about what steps to take. (Substitution, Elimination)

Follow the same steps every time, but do a lot of calculations. (Matrix inverse, Cramer's Rule)

It is also possible to determine the number of solutions—zero, one, or infinity—without actually solving the system.

- Determinant* of a matrix
- Rank of a matrix

* only when # of equations = # of variables

Solving systems using matrices

The system of three equations

$$\begin{cases} 6x + y + 5z = 5 \\ 2y + 9z = 3 \\ -x + 4y + 18z = 5 \end{cases}$$

can be written as the single equation

$$\begin{bmatrix} 6 & 1 & 5 \\ 0 & 2 & 9 \\ -1 & 4 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix}$$

using matrices.

We usually write this as $AX = B$ and call A the **matrix of coefficients**.

Previously: Solving for matrix X

If $AX = B$ and A is invertible then…

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

not BA^{-1}

recall $A^{-1}A = I$

$$IX = A^{-1}B$$

recall $IX = X$

$$X = A^{-1}B$$

Let's also think about

$$3x = 5$$

$$\frac{1}{3}\cdot 3x = \frac{1}{3}\cdot 5$$

same as $5\cdot\frac{1}{3}$

$$1x = \frac{5}{3}$$

$$x = \frac{5}{3}$$

Solving by inverse matrix

Any system of linear eqns corresponds to a single equation $AX = B$. Example:

$$\begin{cases} 5x + 2y - 2z = 4 \\ x - 4z = 2 \\ 12x + 7y + 14z = 5 \end{cases} \quad \to \quad \begin{bmatrix} 5 & 2 & -2 \\ 1 & 0 & -4 \\ 12 & 7 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}.$$

If the coefficient matrix A is invertible, then we can solve the system as

$$X = A^{-1}B.$$

In order to use this, we need to know A^{-1}.

- For 2×2 matrices there is a formula you can memorize.
- For 3×3 and bigger, the steps you could use to solve a system of equations by elimination can also be used to find A^{-1}!

Example: Find the inverse of $\begin{bmatrix} 5 & -2 \\ 1 & 4 \end{bmatrix}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} 5 & -2 \\ 1 & 4 \end{bmatrix}^{-1} = \frac{1}{5(4)-(-2)1}\begin{bmatrix} 4 & 2 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 4/22 & 2/22 \\ -1/22 & 5/22 \end{bmatrix}$$

Check that multiplying A by A^{-1} really gives I:

$$\begin{bmatrix} 5 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 4/22 & 2/22 \\ -1/22 & 5/22 \end{bmatrix} = \begin{bmatrix} 5(\frac{4}{22})+(-2)(\frac{-1}{22}) & 5(\frac{2}{22})+(-2)(\frac{5}{22}) \\ 1(\frac{4}{22})+4(\frac{-1}{22}) & 1(\frac{2}{22})+4(\frac{5}{22}) \end{bmatrix}$$

$$= \begin{bmatrix} 22/22 & 0/22 \\ 0/22 & 22/22 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Eqn with square matrix

Example: solve $\begin{cases} 5x - 2y = 15 \\ x + 4y = 14 \end{cases}$ using an inverse matrix.

$$\begin{bmatrix} 5 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 15 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{4}{22} & \frac{2}{22} \\ \frac{-1}{22} & \frac{5}{22} \end{bmatrix} \begin{bmatrix} 15 \\ 14 \end{bmatrix} = \begin{bmatrix} \frac{4}{22}(15) + \frac{2}{22}(14) \\ \frac{-1}{22}(15) + \frac{5}{22}(14) \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 5/2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}}$$

Augmented matrix

For a system $A\vec{x} = \vec{b}$, the matrix A is called the "coefficient matrix".

The **augmented matrix** for the system is the matrix formed by adding column $\vec{b}$ to the matrix A. We write $\begin{bmatrix} A & \vec{b} \end{bmatrix}$ for this matrix.

- Example: For the system

$$\begin{cases} 4x + 9y = 6 \\ 2x + 3y = 0 \end{cases}$$

we have

$$A = \begin{bmatrix} 4 & 9 \\ 2 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} A & \vec{b} \end{bmatrix} = \begin{bmatrix} 4 & 9 & 6 \\ 2 & 3 & 0 \end{bmatrix}.$$

Often written

$$\left[\begin{array}{cc|c} 4 & 9 & 6 \\ 2 & 3 & 0 \end{array}\right]$$

with a | before the last column.

Row operations

Replacing a row of a matrix with a linear combination of rows is called a **row operation**. This is usually done with augmented matrices.

- By changing $\left[\begin{array}{cc|c} 5 & -2 & 15 \\ 1 & 4 & 14 \end{array}\right]$ into $\left[\begin{array}{cc|c} 1 & 0 & ? \\ 0 & 1 & ? \end{array}\right]$ we solve $\begin{cases} 5x - 2y = 15 \\ x + 4y = 14 \end{cases}$.
- By changing $\left[\begin{array}{cc|cc} 5 & -2 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right]$ into $\left[\begin{array}{cc|cc} 1 & 0 & ? & ? \\ 0 & 1 & ? & ? \end{array}\right]$, we find $\begin{bmatrix} 5 & -2 \\ 1 & 4 \end{bmatrix}^{-1}$.

Example: Find the inverse of the matrix $\begin{bmatrix} 5 & 2 & -2 \\ 1 & 0 & -4 \\ 12 & 7 & 14 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 5 & 2 & -2 & 1 & 0 & 0 \\ 1 & 0 & -4 & 0 & 1 & 0 \\ 12 & 7 & 14 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 0 & 1 & 0 \\ 5 & 2 & -2 & 1 & 0 & 0 \\ 12 & 7 & 14 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 0 & 1 & 0 \\ 0 & 2 & 18 & 1 & -5 & 0 \\ 12 & 7 & 14 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 0 & 1 & 0 \\ 0 & 2 & 18 & 1 & -5 & 0 \\ 0 & 7 & 62 & 0 & -12 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 0 & 1 & 0 \\ 0 & 1 & 9 & \frac{1}{2} & -\frac{5}{2} & 0 \\ 0 & 7 & 62 & 0 & -12 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 0 & 1 & 0 \\ 0 & 1 & 9 & \frac{1}{2} & -\frac{5}{2} & 0 \\ 0 & 0 & 1 & \frac{7}{2} & -\frac{11}{2} & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 0 & 1 & 0 \\ 0 & 1 & 0 & -31 & 47 & 9 \\ 0 & 0 & 1 & \frac{7}{2} & -\frac{11}{2} & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 14 & -21 & -4 \\ 0 & 1 & 0 & -31 & 47 & 9 \\ 0 & 0 & 1 & \frac{7}{2} & -\frac{11}{2} & -1 \end{array}\right]$$

Solving by inverse matrix

From before, we can represent a system of equations as $AX = B$:

$$\begin{cases} 5x + 2y - 2z = 4 \\ x - 4z = 2 \\ 12x + 7y + 14z = 5 \end{cases} \rightarrow \begin{bmatrix} 5 & 2 & -2 \\ 1 & 0 & -4 \\ 12 & 7 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}.$$

If the coefficient matrix A is invertible, then we can solve the system as

$$X = A^{-1}B.$$

Example:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 & 2 & -2 \\ 1 & 0 & -4 \\ 12 & 7 & 14 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 14 & -21 & -4 \\ -31 & 47 & 9 \\ \frac{7}{2} & \frac{-11}{2} & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -6 \\ 15 \\ -2 \end{bmatrix}.$$

There are many methods to solve systems of linear equations by hand.

- Substitution
- Elimination
- Matrix inverse*
- Cramer's Rule*

* only when # of equations = # of variables

Eqn with square matrix

- We can solve the matrix equation $AX = B$ as

$$X = A^{-1}B$$

if we first compute the inverse of the matrix A.

- **Cramer's Rule** is a direct formula for each variable:

$$x_i = \frac{\det(A_i)}{\det(A)},$$

where "A_i" is the matrix formed by replacing $\text{Column } i$ of matrix A with the single column B.

Eqn with square matrix

Example: solve $\begin{cases} 5x - 2y = 15 \\ x + 4y = 14 \end{cases}$ using Cramer's Rule.

$$x = \frac{\det\left(\begin{bmatrix} 15 & -2 \\ 14 & 4 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 5 & -2 \\ 1 & 4 \end{bmatrix}\right)} = \frac{15(4) - (-2)14}{5(4) - (-2)1} = \frac{88}{22} = 4 = x$$

$$y = \frac{\det\left(\begin{bmatrix} 5 & 15 \\ 1 & 14 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 5 & -2 \\ 1 & 4 \end{bmatrix}\right)} = \frac{5(14) - (15)1}{5(4) - (-2)1} = \frac{55}{22} = \frac{5}{2} = y$$

Problems with A-1B and Cramer

These methods are each only possible if

- A has the same number of rows as columns (a "square" matrix)
- and $\det(A) \neq 0$.

Otherwise, A^{-1} does not exist.

If $\det(A) = 0$, the system may or may not have solutions.

- $\det\left(\begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix}\right) = 6(1) - 2(3) = 0.$
- $\begin{cases} 6x + 3y = 15 \\ 2x + y = 5 \end{cases}$ has solutions but $\begin{cases} 6x + 3y = 10 \\ 2x + y = 8 \end{cases}$ does not.

For each system below, ask yourself

- How many variables are there?
- How many equations are there?
- Is there a solution?

$$\begin{cases} x + y = 7 \\ 2x + y = 10 \end{cases}$$

2 variables
2 equations
rank = 2
one solution

$$\begin{cases} x + y = 7 \\ 2x + y = 10 \\ x + y = 7 \end{cases}$$

2 variables
3 equations
rank = 2
one solution

$$\begin{cases} x + y = 7 \\ 2x + y = 10 \\ 3x + 2y = 17 \end{cases}$$

2 variables
3 equations
rank = 2
one solution

$$\begin{cases} x + y = 7 \\ 2x + y = 10 \\ 3x + 2y = 18 \end{cases}$$

2 variables
3 equations
rank = 3
no solutions

Related topics

Linear combinations of vectors

Linearly independent* collections of vectors

Systems of linear equations: the collection of all solutions can form

- nothing (no solution)
- a single point
- a line
- a plane
- a “hyperplane” (if you have 4 or more variables)

The rank* of the coefficient matrix helps determine which.

* We will define these soon.

Linear combinations

A **linear combination** of some vectors is any sum of scalar multiples of those vectors.

- In symbols, $\vec{u}$ is a linear combination of $\vec{v}$ and $\vec{w}$ if
$$\vec{u} = s\vec{v} + t\vec{w}$$
for some numbers s, t.

- For more vectors, $\vec{u}$ is a linear combination of $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n}$ if
$$\vec{u} = s_1\overrightarrow{v_1} + s_2\overrightarrow{v_2} + \cdots + s_n\overrightarrow{v_n}$$
for some numbers (scalars) $s_1, \ldots, s_n$.

Linear combinations

Previ-ously

A **linear combination** of some vectors is any sum of scalar multiples of those vectors.

- In symbols, $\vec{u}$ is a linear combination of $\vec{v}$ and $\vec{w}$ if
$$\vec{u} = s\vec{v} + t\vec{w}$$
for some numbers s, t.

Example 1: Write $\begin{bmatrix} 5 \\ 24 \end{bmatrix}$ as a linear combination of $\vec{v_1} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 3 \\ -9 \end{bmatrix}$.

We want $x\begin{bmatrix} 5 \\ -2 \end{bmatrix} + y\begin{bmatrix} 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 5 \\ 24 \end{bmatrix}$, so we must solve the system $\begin{cases} 5x+3y=5 \\ -2x-9y=24 \end{cases}$.

Solution: $x = 3$, $y = -10/3$. Therefore $\begin{bmatrix} 5 \\ 24 \end{bmatrix} = 3\begin{bmatrix} 5 \\ -2 \end{bmatrix} + (-10/3)\begin{bmatrix} 3 \\ -9 \end{bmatrix}$.

Linear dependence

We can use any of these three definitions:

- A collection of vectors is called **linearly dependent** (or **LD**) if one vector *is* a linear combination of the others.

- A collection $\overrightarrow{v_1}, \ldots, \overrightarrow{v_n}$ is **linearly dependent** if there exist numbers $s_1, \ldots, s_n$ not *all* zero such that

$$s_1\overrightarrow{v_1} + s_2\overrightarrow{v_2} + \cdots + s_n\overrightarrow{v_n} = \vec{0}.$$

  - Note: *some* of the s_i can be zero, just not all.

- A collection is **linearly dependent** if it is not linearly independent.

Linear independence

We can use any of these three definitions:

- A collection of vectors is called **linearly independent** if no vector is a linear combination of the others.

- A collection $\overrightarrow{v_1}, \ldots, \overrightarrow{v_n}$ is **linearly independent** if the *only* solution to

$$s_1\overrightarrow{v_1} + s_2\overrightarrow{v_2} + \cdots + s_n\overrightarrow{v_n} = \vec{0}$$

is $s_1 = s_2 = \cdots = s_n = 0$.

- A collection is **linearly independent** if it is not linearly dependent.

Linear (in)dependence

Note that a single vector isn't called linearly dependent or independent. This is about collections of vectors.

- Example: $\left\{ \begin{bmatrix} 5 \\ 24 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is linearly dependent.

However, it is common to skip the $\{ \}$ and talk about the vectors directly.

- Example: "$\begin{bmatrix} 5 \\ 24 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are linearly dependent." is just a lazy way of saying the first bullet.

Linear (in)dependence

- Example: Determine whether

$$\vec{u} = \begin{bmatrix} -1 \\ 5 \\ 7 \end{bmatrix}, \ \vec{v} = \begin{bmatrix} 4 \\ -2 \\ 12 \end{bmatrix}, \ \vec{w} = \begin{bmatrix} -1 \\ 14 \\ 27 \end{bmatrix}$$

are linearly dependent or linearly independent.

Dependent. Using the first definition, this is because $\vec{w} = 3\vec{u} + \frac{1}{2}\vec{v}$, or because $\vec{v} = -6\vec{u} + 2\vec{w}$, etc.

Using the second definition, this is because, e.g.,

$$6\vec{u} + \vec{v} + (-2)\vec{w} = [0,0,0].$$

Linear (in)dependence

Some facts to notice:

If a collection contains the zero vector then it is linearly dependent.

If the vectors are d-dimensional (each is a list of d numbers), then any collection of $d+1$ or more vectors will be linearly dependent.

Examples:

- $\left\{ \begin{bmatrix} 1 \\ -9 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 7 \end{bmatrix} \right\}$ must be LD. Note $0\overrightarrow{v_1} + 5\overrightarrow{v_2} + 0\overrightarrow{v_3} = \vec{0}$.
- $\left\{ \begin{bmatrix} 3 \\ -8 \end{bmatrix}, \begin{bmatrix} 5 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ must be LD.

Rank

The **rank** of a matrix is the maximum number of linearly independent rows.

- Remember that a set of vectors is **linearly independent** if no vector is a linear combination of the others.
  - Remember that a **linear combination** of vectors is any sum of scalar multiples of the vectors: $a\vec{v} + b\vec{w} + \cdots$

max. # of lin. indep. rows = max. # of lin. indep. columns

An $n \times m$ matrix can have rank at most $\min(n, m)$.
An $n \times m$ matrix is called **full rank** if its rank is equal to $\min(n, m)$.

Rank

The **rank** of a matrix is the maximum number of linearly independent rows (and also the maximum number of linearly independent columns—these will always be the same number!).

Example: What is the rank of $\begin{bmatrix} 3 & 1 & -9 & 0 & 6 \\ 2 & 0 & 4 & 1 & -3 \end{bmatrix}$? rank 2

Example: What is the rank of $\begin{bmatrix} -9 & 18 \\ 2 & -4 \\ 5 & -10 \end{bmatrix}$? rank 1

because $\begin{bmatrix} 18 \\ -4 \\ -10 \end{bmatrix} = -2 \begin{bmatrix} -9 \\ 2 \\ 5 \end{bmatrix}$

Rank

The **rank** of a matrix is the maximum number of linearly independent rows (and also the maximum number of linearly independent columns—these will always be the same number!).

Example: What is the rank of $\begin{bmatrix} 5 & 2 & 7 \\ 1 & 0 & 1 \\ 12 & 7 & 19 \end{bmatrix}$?

rank 2

because $\begin{bmatrix} 7 \\ 1 \\ 19 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 12 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 7 \end{bmatrix}$

Example: What is the rank of $\begin{bmatrix} 5 & 2 & 7 \\ 1 & 0 & 1 \\ 12 & 7 & 18 \end{bmatrix}$?

rank 3

because $\begin{bmatrix} 7 \\ 1 \\ 19 \end{bmatrix} = a\begin{bmatrix} 5 \\ 1 \\ 12 \end{bmatrix} + b\begin{bmatrix} 2 \\ 0 \\ 7 \end{bmatrix}$ is impossible

Systems – summary so far

Any system of linear equations can be written as

$$A\vec{x} = \vec{b}.$$

coefficients — variables — right-hand side

- If A is square (same # of rows and cols) *and* $\det(A) \neq 0$, then the inverse matrix A^{-1} exists and the system has exactly one solution:

$$\vec{x} = A^{-1}\vec{b}.$$

- If A is square but $\det(A) = 0$, the system has either 0 or infinitely many solutions.
- If A is not square, there is no determinant or inverse.

rank(A) will help us determine the number of solutions in these cases.

Rank as amount of information

Suppose we know that $\begin{cases} x+y+z=6 \\ x+y \quad\quad = 3. \end{cases}$

Can we stay anything about $x+y-3z$?

×4 ×(−3)

$$\begin{array}{rl} 4x + 4y & = 12 \\ +\ (-3x - 3y - 3z & = -18) \\ \hline x + y - 3z & = -6 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 1 & 0 & 3 \end{bmatrix}$$

has rank 2

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 1 & 0 & 3 \\ 1 & 1 & -3 & -6 \end{bmatrix}$$

also has rank 2

No new information!

Rank as amount of information

$$\begin{cases} x+y+z=6 \\ x+y \qquad\;\; =3 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 1 & 0 & 3 \end{bmatrix}$$

has rank 2

$$\begin{cases} x+y+\;\; z=6 \\ x+y \qquad\;\; =3 \\ x+y-3z=-6 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 1 & 0 & 3 \\ 1 & 1 & -3 & -6 \end{bmatrix}$$

has rank 2

$$\begin{cases} x+y+\;\; z=6 \\ x+y \qquad\;\; =3 \\ x+5y-z=5 \end{cases}$$

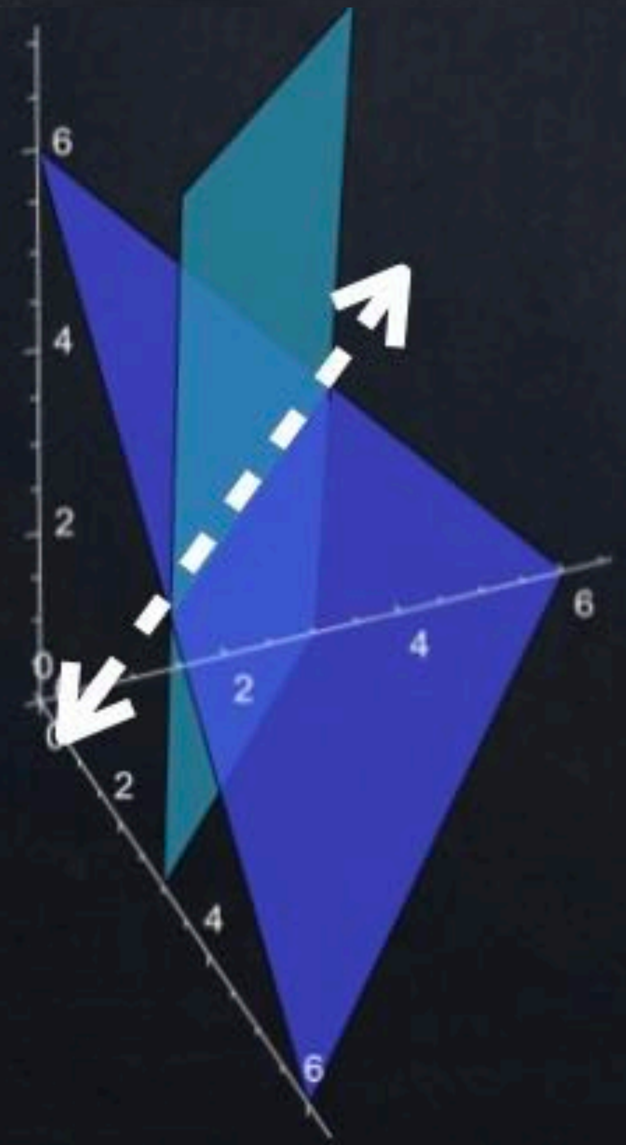
$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 1 & 0 & 3 \\ 1 & 5 & -1 & 5 \end{bmatrix}$$

has rank 3

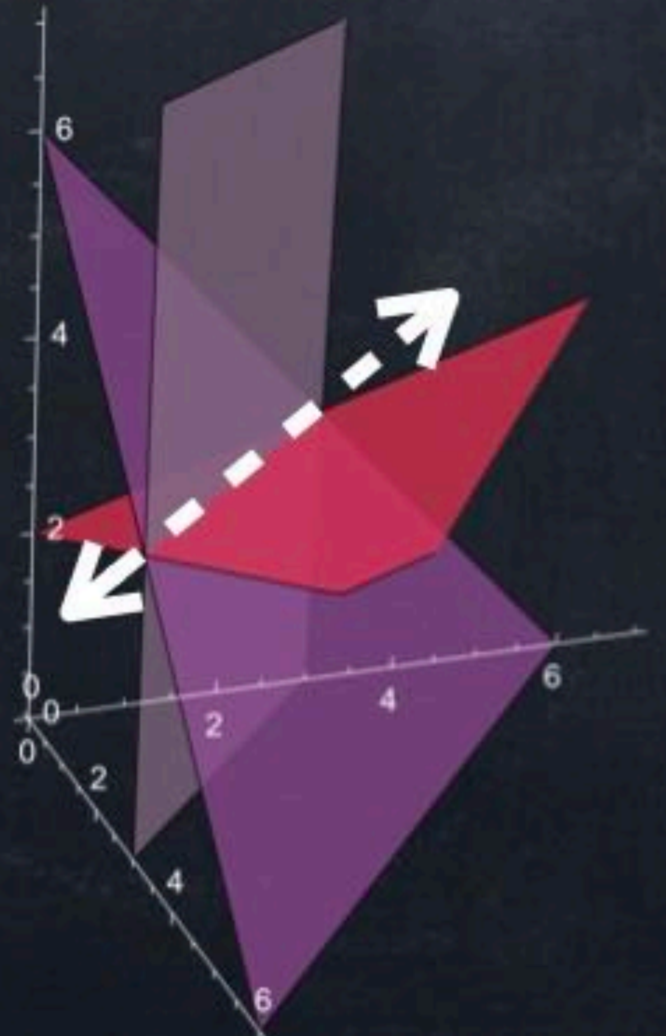
$$\begin{cases} x+y+\;\; z=6 \\ x+y \qquad\;\; =3 \\ 5x+5y \qquad =10 \end{cases}$$

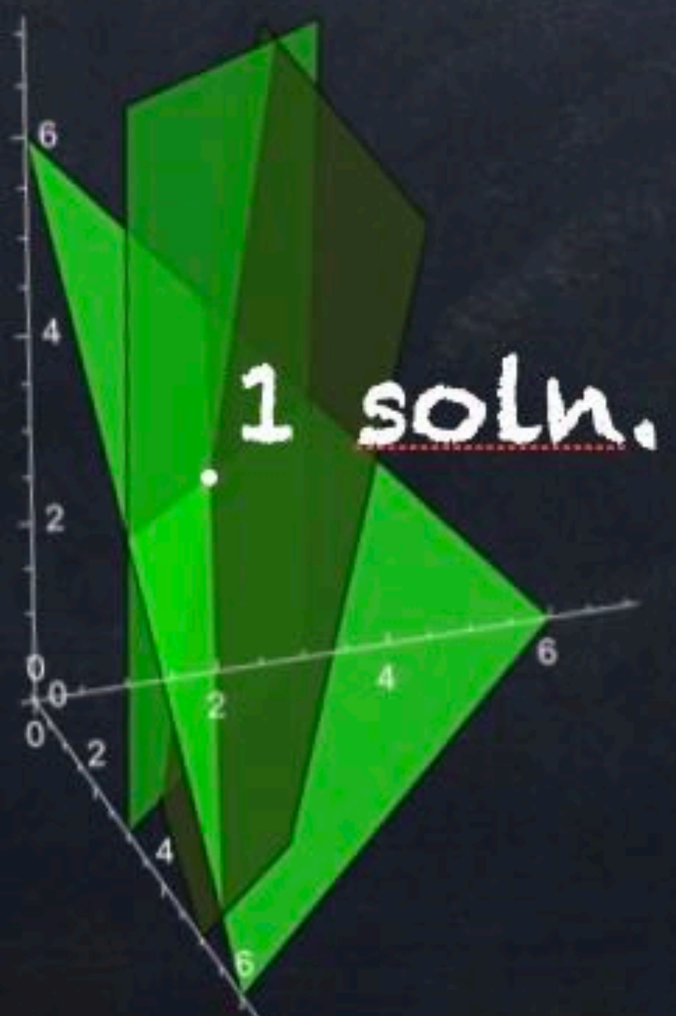
$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 1 & 0 & 3 \\ 5 & 5 & 0 & 10 \end{bmatrix}$$

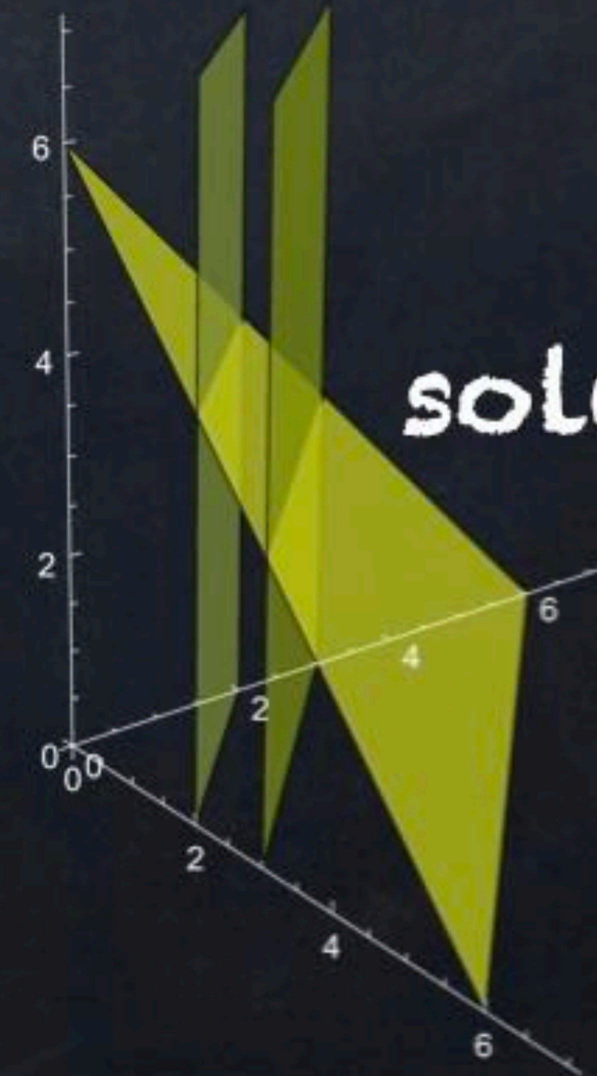
has rank 3



infinitely many solutions







no solutions

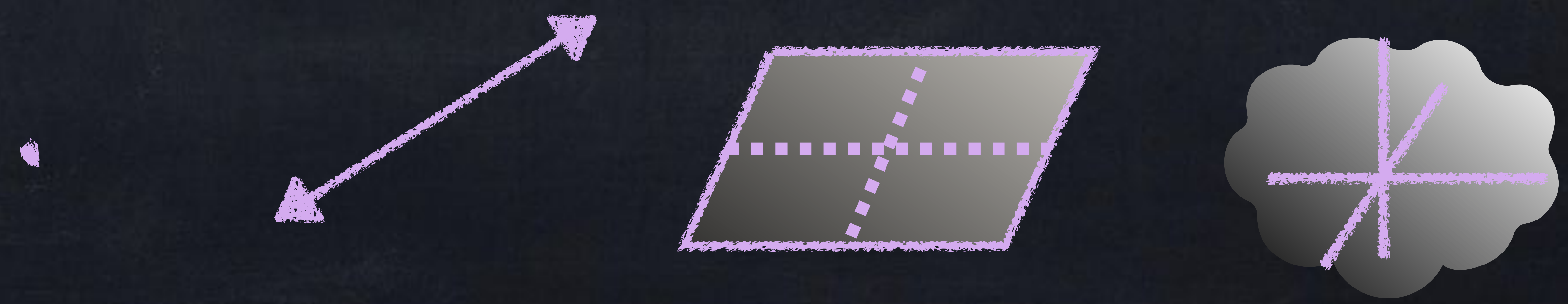
The Rouché–Capelli Theorem

The system $A\vec{x} = \vec{b}$ has at least one solution
if and only if $\mathrm{rank}(A) = \mathrm{rank}\big([A \;\; \vec{b}]\big)$.

Reminder: A is the "coefficient matrix", and $[A \;\; \vec{b}]$ is the "augmented matrix".

If there are any solutions, the collection of all solutions has dimension $n - \mathrm{rank}(A)$, where n is the number of variables.

Dimension: 0 1 2 3



Math 1433

4 December 2023

Warm-up:

Find the determinant of $A = \begin{bmatrix} 5 & 2 & 7 \\ 1 & 0 & 1 \\ 12 & 7 & 18 \end{bmatrix}$.

Rank/system examples

Ex 1.

$$\begin{cases} 5x + 2y + 7z = 6 \\ x + z = 4 \\ 12x + 7y + 18z = 9 \end{cases}$$

$$A = \begin{bmatrix} 5 & 2 & 7 \\ 1 & 0 & 1 \\ 12 & 7 & 18 \end{bmatrix}$$

$$[A \ \vec{b}] = \left[\begin{array}{ccc|c} 5 & 2 & 7 & 6 \\ 1 & 0 & 1 & 4 \\ 12 & 7 & 18 & 9 \end{array}\right]$$

rank(A) = 3

rank(A|b) = 3

The coeff. and augmented matrices have the same rank, so the system does have at least one solution. The space of all solutions has dimension

(# of variables) - (rank of A) = 3 - 3 = 0,

so the set of solutions is just one point.

Rank/system examples

Ex 2.

$$\begin{cases} 5x + 2y + 7z = 6 \\ x + z = 4 \\ 12x + 7y + 19z = 9 \end{cases}$$

$$A = \begin{bmatrix} 5 & 2 & 7 \\ 1 & 0 & 1 \\ 12 & 7 & 19 \end{bmatrix}$$

$$[A\ \vec{b}] = \left[\begin{array}{ccc|c} 5 & 2 & 7 & 6 \\ 1 & 0 & 1 & 4 \\ 12 & 7 & 19 & 9 \end{array}\right]$$

rank(A) = 2

rank(A|b) = 3

The coefficient and augmented matrices have different ranks, so there are no solutions to the system.

Rank/system examples

Ex 3.

$$\begin{cases} 5x + 2y + 7z = \underline{10} \\ x + z = \underline{4} \\ 12x + 7y + 19z = \underline{13} \end{cases}$$

$$A = \begin{bmatrix} 5 & 2 & 7 \\ 1 & 0 & 1 \\ 12 & 7 & 19 \end{bmatrix}$$

$$[A \ \vec{b}] = \left[\begin{array}{ccc|c} 5 & 2 & 7 & \underline{10} \\ 1 & 0 & 1 & \underline{4} \\ 12 & 7 & 19 & \underline{13} \end{array}\right]$$

rank(A) = 2

rank(A|b) = 2

The coeff. and augmented matrices have the same rank, so the system does have at least one solution. The space of all solutions has dimension

(# of variables) - (rank of A) = 3 - 2 = 1,

so the set of solutions is a LINE in 3D space.