Linear Algebra, Winter 2022

List 1

Algebra and trig review, vector operations

1. Which of the following are true for **all** real values of the variables?
   (a) $2x = x + x$ $\boxed{\text{True}}$
   (b) $2(x + y) = 2x + y$ $\boxed{\text{False}}$
   (c) $(x - y)^2 = x^2 - 2xy + y^2$ $\boxed{\text{True}}$
   (d) $(6 + a)/2 = 3 + a/2$ $\boxed{\text{True}}$
   (e) $-(y + 2) = -y + 2$ $\boxed{\text{False}}$
   (f) $-(a + b)^2 = (-a + b)^2$ $\boxed{\text{False}}$
   (g) $x^3 + 3x = x + x$ $\boxed{\text{False}}$
   (h) $k^{-2} = 1/k^2$ $\boxed{\text{True}}$
   (i) $k^{-2} = -\sqrt{k}$ $\boxed{\text{False}}$
   (j) $x^{a+2} = x^a \times x^2$ $\boxed{\text{True}}$

2. Compute the following values:
   (a) $\cos(0)$ $\boxed{1}$
   (b) $\sin(0)$ $\boxed{0}$
   (c) $\cos(30°)$ $\boxed{\sqrt{3}/2}$
   (d) $\cos(45°)$ $\boxed{1/\sqrt{2}}$ or $\boxed{\sqrt{2}/2}$
   (e) $\cos(60°)$ $\boxed{1/2}$
   (f) $\cos(\pi/3)$ $\boxed{1/2}$ (same as previous)
   (g) $\cos(\pi/2)$ $\boxed{0}$
   (h) $\sin(\pi/2)$ $\boxed{1}$
   (i) $\sin(180°)$ $\boxed{0}$
   (j) $\sin(4\pi/3)$ $\boxed{-\sqrt{3}/2}$
   (k) $\arccos(\frac{1}{\sqrt{2}})$ $\boxed{\pi/4}$ or $\boxed{45°}$
   (l) $\arccos(\frac{\sqrt{3}}{2})$ $\boxed{\pi/6}$ or $\boxed{30°}$

3. Find *one* value of θ for which both $-\sqrt{5} = \sqrt{20}\cos(\theta)$ and $\sqrt{15} = \sqrt{20}\sin(\theta)$.
   $\boxed{\dfrac{2\pi}{3}}$ or $\dfrac{2\pi}{3} + 2\pi n$ for any integer n

4. Simplify $(2\,e^7)^{10}$. $\boxed{1024\,e^{70}}$

In 2D, the **zero vector** is $\vec{0} = [0,0]$, and the **standard basis vectors** are $\hat{\imath} = [1,0]$ and $\hat{\jmath} = [0,1]$. In 3D, the **zero vector** is $\vec{0} = [0,0,0]$, and the **standard basis vectors** are $\hat{\imath} = [1,0,0]$ and $\hat{\jmath} = [0,1,0]$ and $\hat{k} = [0,0,1]$.

In any dimension,

magnitude (or **length**): $\left|[a_1, ..., a_n]\right| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$

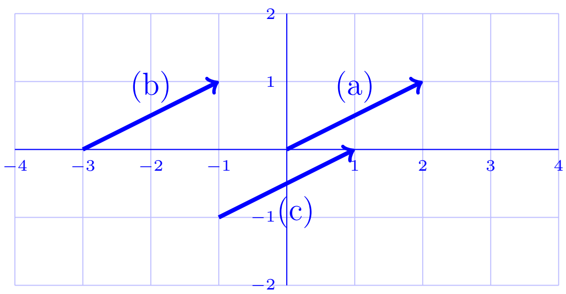
scalar multiplication: $s[a_1, ..., a_n] = [sa_1, ..., sa_n]$

vector addition: $[a_1, ..., a_n] + [b_1, ..., b_n] = [a_1+b_1, ..., a_n+b_n]$

vector subtraction: $[a_1, ..., a_n] - [b_1, ..., b_n] = [a_1-b_1, ..., a_n-b_n]$

5. Calculate each of the following:
   (a) $[3,2] + [7,1]$ $\boxed{[10,3]}$ or $\boxed{\left[\begin{smallmatrix}10\\3\end{smallmatrix}\right]}$ or $\boxed{10\hat{\imath} + 3\hat{\jmath}}$
   (b) $\langle 3,2\rangle + \langle 7,1\rangle$ same as part (a)
   (c) $5[-4,3] = \boxed{[-20,15]}$
   (d) $8\begin{pmatrix}3\\2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}7\\1\end{pmatrix} = \begin{pmatrix}24\\16\end{pmatrix} + \begin{pmatrix}3.5\\0.5\end{pmatrix} = \begin{pmatrix}27.5\\16.5\end{pmatrix}$ or $\boxed{\begin{pmatrix}55/2\\33/2\end{pmatrix}}$
   (e) $\frac{1}{20}[3,2] = \boxed{\left[\frac{3}{20}, \frac{1}{10}\right]}$
   (f) $9[1,0] + 2[0,1] = \boxed{[9,2]}$
   (g) $9\hat{\imath} + 2\hat{\jmath}$ (in 2D) $= \boxed{[9,2]}$
   (h) $6\hat{\imath} + \hat{\jmath} - 2\hat{k} = \boxed{[6,1,-2]}$
   (i) $6\hat{\jmath} - 4(\hat{\jmath} - \hat{k}) = \boxed{[0,2,4]}$

6. **Draw** the following vectors as arrows all on the same plane (one drawing, not three drawings):
   (a) the vector $2\hat{\imath} + \hat{\jmath}$ with its start at $(0,0)$.
   (b) the vector $2\hat{\imath} + \hat{\jmath}$ with its start at $(-3,0)$.
   (c) the vector $2\hat{\imath} + \hat{\jmath}$ with its start at $(-1,-1)$.



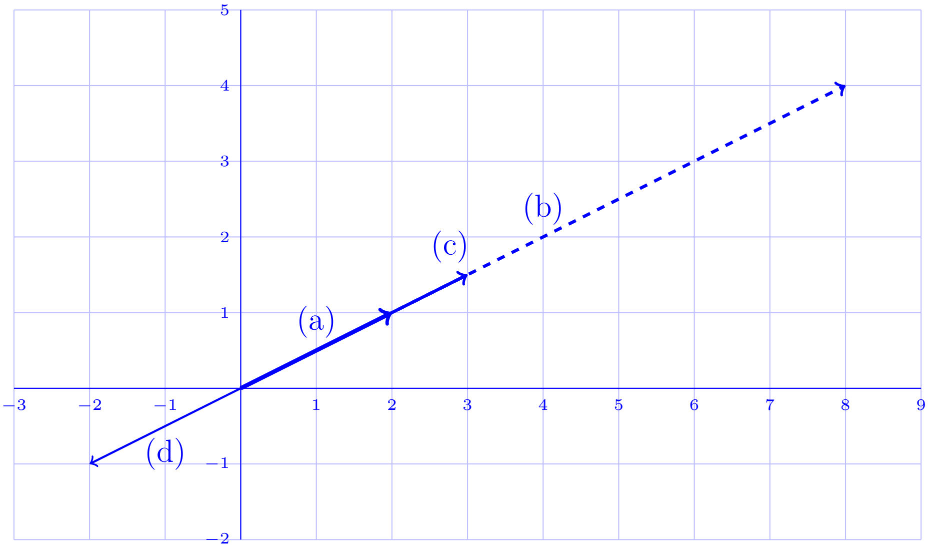
7. Draw the following vectors as arrows all on the same plane (one drawing, not four drawings):

(a) the vector $[2, 1]$ with its start at $(0, 0)$.

(b) the vector $4[2, 1]$ with its start at $(0, 0)$.

(c) the vector $1.5[2, 1]$ with its start at $(0, 0)$.

(d) the vector $(-1)[2, 1]$ with its start at $(0, 0)$.



8. Which of the following are scalar multiples of $[4, 2, -6]$?

   (a) $\begin{pmatrix} 20 \\ 10 \\ -60 \end{pmatrix}$ No

   (b) $[-12, -6, 18]$ Yes

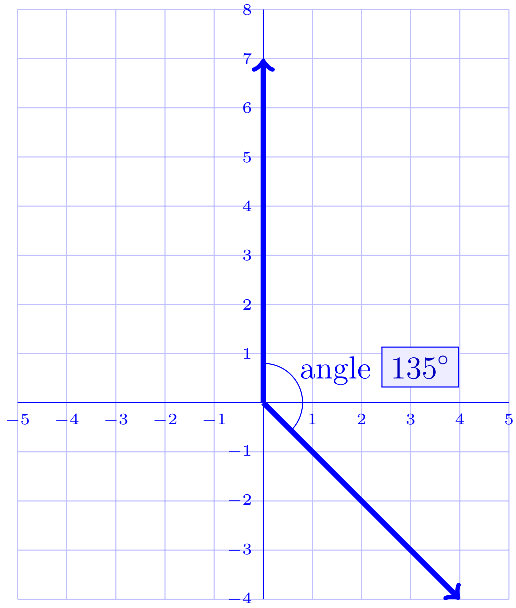
   (c) $[0, 0, 0]$ Yes

   (d) $\begin{pmatrix} 0.4 \\ 0.2 \\ -0.6 \end{pmatrix}$ Yes

   (e) $\begin{pmatrix} \sqrt{32} \\ \sqrt{8} \\ -\sqrt{72} \end{pmatrix}$ Yes

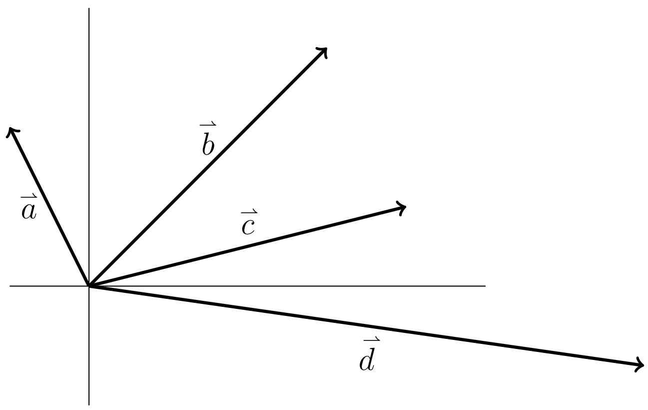
   (f) $[8, 4, -10]$ No

9. Draw, on one picture, the vectors $7\hat{\jmath}$ and $4\hat{\imath} - 4\hat{\jmath}$ as arrows starting at $(0, 0)$. What is the angle between these two vectors?



10. Let P be the point $(5,2)$ and let Q be the point $(1,9)$. Describe the vector $\begin{bmatrix}5\\2\end{bmatrix} - \begin{bmatrix}1\\9\end{bmatrix}$ in words, *without doing any calculations.* An arrow from Q to P

11. Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be as in the image below.



Write a true equation of the form __ + __ = __ using these vectors.
$\vec{a} + \vec{c} = \vec{b}$ or $\vec{c} + \vec{a} = \vec{b}$.

12. Write two true equations of the form __ − __ = __ using vectors from Task 11.
$\vec{b} - \vec{a} = \vec{c}$ and $\vec{b} - \vec{c} = \vec{a}$.

13. Calculate the following vectors or scalars:

(a) $|4\hat{\imath} - 4\hat{\jmath}| = \sqrt{4^2 + (-4)^2} = \sqrt{16+16} = \boxed{\sqrt{32} \text{ or } 4\sqrt{2}}$

(b) $|[0,7]| = \boxed{7}$

(c) $\dfrac{[3,2]}{|[3,2]|} = \boxed{\left[\dfrac{3}{\sqrt{13}}, \dfrac{2}{\sqrt{13}}\right]}$

(d) $\left|\dfrac{[3,2]}{|[3,2]|}\right| = \boxed{1}$

14. Calculate each of the following. Each answer will be either a scalar *formula* or a vector *formula* involving t.

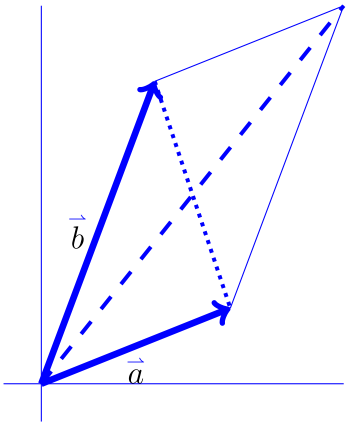
(a) $5\begin{pmatrix}3\\2\end{pmatrix}+t\begin{pmatrix}7\\1\end{pmatrix} = \boxed{\begin{pmatrix}15+7t\\10+t\end{pmatrix}}$

(b) $t+\left|\left[\begin{smallmatrix}3\\2\end{smallmatrix}\right]\right| = \boxed{t+\sqrt{13}}$

(c) $\left|t[3,2]\right| = \boxed{t\sqrt{13}}$

(d) $\left|[1+t,1-t]\right|^2 = \left(\sqrt{(1+t)^2+(1-t)^2}\right)^2 = (1+t)^2+(1-t)^2$ or $\boxed{2+2t^2}$

15. A parallelogram has the vector $\vec{a}=[5,2]$ along one edge and $\vec{b}=[3,8]$ along another edge. Compute the lengths of the two diagonals of the parallelogram.



The longer diagonal (dashed) is $\vec{a}+\vec{b}$. Its length is

$$\left|\vec{a}+\vec{b}\right| = \left|[5+3,2+8]\right| = \left|[8,10]\right| = \sqrt{8^2+10^2} = \boxed{\sqrt{164}}$$

The shorter diagonal (dotted) is $\vec{a}-\vec{b}$. Its length is

$$\left|\vec{a}-\vec{b}\right| = \left|[5-3,2-8]\right| = \left|[2,-6]\right| = \sqrt{2^2+(-6)^2} = \boxed{\sqrt{40}}$$

16. Give a vector that is parallel to $\vec{v}=[8,-1,4]$ but has length 1.

$$\frac{[8,-1,4]}{\left|[8,-1,4]\right|} = \frac{[8,1,4]}{\sqrt{81}} = \boxed{\left[\frac{8}{9},\frac{-1}{9},\frac{4}{9}\right]}$$

The **dot product** (also called **scalar product**) of $\vec{u}$ and $\vec{v}$ is written $\vec{u}\cdot\vec{v}$ and can be calculated as either

$$\vec{u}\cdot\vec{v} = u_1v_1+u_2v_2+\cdots+u_nv_n$$

or

$$\vec{u}\cdot\vec{v} = \left|\vec{u}\right|\left|\vec{v}\right|\cos(\text{angle between } \vec{u} \text{ and } \vec{v}).$$

Two vectors are called **orthogonal** if their dot product is 0.

17. Calculate $\begin{pmatrix}5\\7\end{pmatrix}\cdot\begin{pmatrix}-8\\1\end{pmatrix}$. $5(-8)+7(1) = -40+7 = \boxed{-33}$

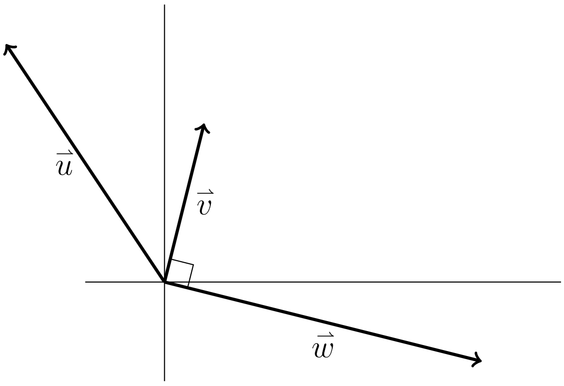
18. Calculate $(4\hat{\imath}-4\hat{\jmath})\cdot(7\hat{\jmath})$ in two ways:

(a) by using $(4)(0)+(-4)(7)$. $0-28=\boxed{-28}$

(b) by using $(4\sqrt{2})(7)\cos(135°)$. (See 13(a), 13(b), and 9.) $(4\sqrt{2})(7)(\frac{-1}{\sqrt{2}}) = \boxed{-28}$

19. Find the angle between $\begin{pmatrix}-3\\\sqrt{3}\end{pmatrix}$ and $\begin{pmatrix}\sqrt{3}\\1\end{pmatrix}$. $\boxed{\frac{2}{3}\pi = 120°}$

20. Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be as in the image below.



(a) Is $\vec{u}\cdot\vec{v}$ positive, negative, or zero? positive

(b) Is $\vec{u}\cdot\vec{w}$ positive, negative, or zero? negative

(c) Is $\vec{v}\cdot\vec{w}$ positive, negative, or zero? zero

21. For each formula below, state whether it represents a scalar (number), a vector, or "nonsense" (meaning it is not a legal operation; for example, $\vec{v}+5$ is nonsense.

(a) $\vec{a}+\vec{b}$ vector

(b) $\vec{u}\cdot\vec{v}$ scalar

(c) $\vec{a}\vec{b}$ nonsense

(d) $t\vec{a}$ vector

(e) $t+\vec{v}$ nonsense

(f) $(t+s)\vec{u}$ vector

(g) $\vec{n}/s$ vector

(h) $\vec{a}-s$ nonsense

(i) $\vec{c}+s\vec{b}$ vector

(j) $t(\vec{a}+\vec{b})-\vec{c}$ vector

(k) $(\vec{a}\cdot\vec{b})\vec{c}$ vector

(l) $\vec{0}-\vec{a}$ vector

(m) $\vec{0}\cdot\vec{w}$ scalar

(n) $\begin{pmatrix}4\\2\end{pmatrix}\cdot\begin{pmatrix}1\\8\end{pmatrix}$ scalar

(o) $[4,2]\cdot[s,t]$ scalar

(p) $\vec{w}\cdot[s,t]$ scalar

(q) $|\vec{u}|$ scalar

(r) $\left|[9,2,\frac{1}{2}]\right|$ scalar

(s) $|\vec{w}|\vec{v}$ vector

(t) $|\vec{a}|+(\vec{b}\cdot\vec{c})$ scalar

(u) $|\vec{a}|(\vec{b}\cdot\vec{c})$ scalar

(v) $(\vec{a})^2$ nonsense

(w) $|\vec{a}|^2$ scalar

22. State whether each pair of vectors below is parallel, perpendicular, or neither.

(a) $\begin{pmatrix}5\\2\end{pmatrix}$ and $\begin{pmatrix}-2\\7\end{pmatrix}$ neither

(b) $\begin{pmatrix}5\\2\end{pmatrix}$ and $\begin{pmatrix}1\\0.4\end{pmatrix}$ parallel

(c) $2\hat{\imath}-8\hat{\jmath}$ and $-8\hat{\imath}+2\hat{\jmath}$ neither

(d) $\begin{pmatrix}10\\-6\\3\end{pmatrix}$ and $\begin{pmatrix}0\\2\\4\end{pmatrix}$ perpendicular

(e) $9\hat{\imath}+11\hat{\jmath}-29\hat{k}$ and $2\hat{\imath}+j+\hat{k}$ perpendicular

(f) $32\hat{\imath}+180\hat{\jmath}$ and $32\hat{\imath}+7\hat{k}$ neither

23. Give an example of a vector that is perpendicular to $\vec{v}=\hat{\imath}+9\hat{\jmath}+4\hat{k}$.

There are many, many correct answers. One correct answer is $[4,0,-1]$.

24. Knowing that

$$\cos(19.5^\circ) \approx \tfrac{33}{35}, \quad \cos(25.2^\circ) \approx \tfrac{19}{21}, \quad \cos(31^\circ) \approx \tfrac{6}{7}, \quad \cos(62.96^\circ) \approx \tfrac{15}{33},$$

find the acute angle between $[6, 3, 6]$ and $[6, 9, 18]$.

We know

$$[6, 3, 6] \cdot [6, 9, 18] = 6(6) + 3(9) + 6(18) = 171$$

and also

$$[6, 3, 6] \cdot [6, 9, 18] = \big|[6, 3, 6]\big|\,\big|[6, 9, 18]\big| \cos\theta = (9)(21)\cos\theta = 189\cos\theta,$$

so it must be that

$$171 = 189\cos\theta \qquad \Rightarrow \qquad \cos\theta = \frac{171}{189} = \frac{19}{21}$$

and therefore $\theta = 25.2^\circ$.

A **linear combination** of a collection of vectors is a sum (+) of scalar multiples of those vectors. (A linear combination of one vector is simply a scalar multiple of that one vector.)

25. Write $\begin{pmatrix} 13 \\ 3 \end{pmatrix}$ as a linear combination of $\hat{\imath} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\hat{\jmath} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. $\boxed{13\hat{\imath} + 3\hat{\jmath}}$

26. Write $\begin{pmatrix} 13 \\ 3 \end{pmatrix}$ as a linear combination of $\vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. $\boxed{3\vec{a} + 5\vec{b}}$

27. Why is it impossible to write $\begin{pmatrix} 13 \\ 3 \end{pmatrix}$ as a linear combination of $\vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{c} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$?

If $\begin{pmatrix} 13 \\ 3 \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ then, from the first components (top rows) we need $13 = x + 3y$ and from the second components (bottom rows) we need $3 = x + 3y$. It is impossible to for $x + 3y$ to equal both 13 and 3.

28. For the vectors

$$\vec{v}_1 = 2\hat{\imath} + 9\hat{\jmath} - 6\hat{k}, \qquad \vec{v}_2 = 4\hat{\imath} + 2\hat{\jmath} - 6\hat{k}, \qquad \vec{v}_3 = -8\hat{\jmath} + 3\hat{k},$$

either write $\vec{v}_1$ as a a linear combination of the other vectors or explain why it is not possible to do so. $\boxed{\vec{v}_1 = \tfrac{1}{2}\vec{v}_2 + (-1)\vec{v}_3}$