Linear Algebra, Winter 2022

List 1

Algebra and trig review, vector operations

1. Which of the following are true for **all** real values of the variables?

   (a) $2x = x + x$

   (b) $2(x + y) = 2x + y$

   (c) $(x - y)^2 = x^2 - 2xy + y^2$

   (d) $(6 + a)/2 = 3 + a/2$

   (e) $-(y + 2) = -y + 2$

   (f) $-(a + b)^2 = (-a + b)^2$

   (g) $x^3 + 3x = x + x$

   (h) $k^{-2} = 1/k^2$

   (i) $k^{-2} = -\sqrt{k}$

   (j) $x^{a+2} = x^a \times x^2$

2. Compute the following values:

   (a) $\cos(0)$

   (b) $\sin(0)$

   (c) $\cos(30°)$

   (d) $\cos(45°)$

   (e) $\cos(60°)$

   (f) $\cos(\pi/3)$

   (g) $\cos(\pi/2)$

   (h) $\sin(\pi/2)$

   (i) $\sin(180°)$

   (j) $\sin(4\pi/3)$

   (k) $\arccos(\frac{1}{\sqrt{2}})$

   (l) $\arccos(\frac{\sqrt{3}}{2})$

3. Find *one* value of θ for which both $-\sqrt{5} = \sqrt{20}\cos(\theta)$ and $\sqrt{15} = \sqrt{20}\sin(\theta)$.

4. Simplify $(2\,e^7)^{10}$.

In 2D, the **zero vector** is $\vec{0} = [0, 0]$, and the **standard basis vectors** are $\hat{\imath} = [1, 0]$ and $\hat{\jmath} = [0, 1]$. In 3D, the **zero vector** is $\vec{0} = [0, 0, 0]$, and the **standard basis vectors** are $\hat{\imath} = [1, 0, 0]$ and $\hat{\jmath} = [0, 1, 0]$ and $\hat{k} = [0, 0, 1]$.

In any dimension,

magnitude (or **length**): $\big|[a_1, ..., a_n]\big| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$

scalar multiplication: $s[a_1, ..., a_n] = [sa_1, ..., sa_n]$

vector addition: $[a_1, ..., a_n] + [b_1, ..., b_n] = [a_1+b_1, ..., a_n+b_n]$

vector subtraction: $[a_1, ..., a_n] - [b_1, ..., b_n] = [a_1-b_1, ..., a_n-b_n]$

5. Calculate each of the following:

   (a) $[3, 2] + [7, 1]$

   (b) $\langle 3, 2\rangle + \langle 7, 1\rangle$

   (c) $5[-4, 3]$

   (d) $8\begin{pmatrix}3\\2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}7\\1\end{pmatrix}$

   (e) $\frac{1}{20}[3, 2]$

   (f) $9[1, 0] + 2[0, 1]$

   (g) $9\hat{\imath} + 2\hat{\jmath}$ (in 2D)

   (h) $6\hat{\imath} + \hat{\jmath} - 2\hat{k}$

   (i) $6\hat{\jmath} - 4(\hat{\jmath} - \hat{k})$

6. **Draw** the following vectors as arrows all on the same plane (one drawing, not three drawings):

   (a) the vector $2\hat{\imath} + \hat{\jmath}$ with its start at $(0, 0)$.

   (b) the vector $2\hat{\imath} + \hat{\jmath}$ with its start at $(-3, 0)$.

   (c) the vector $2\hat{\imath} + \hat{\jmath}$ with its start at $(-1, -1)$.

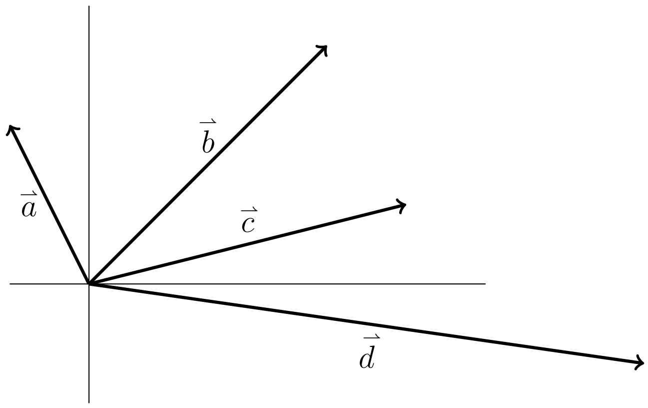
7. Draw the following vectors as arrows all on the same plane (one drawing, not four drawings):
   (a) the vector $[2,1]$ with its start at $(0,0)$.
   (b) the vector $4[2,1]$ with its start at $(0,0)$.
   (c) the vector $1.5[2,1]$ with its start at $(0,0)$.
   (d) the vector $(-1)[2,1]$ with its start at $(0,0)$.

8. Which of the following are scalar multiples of $[4,2,-6]$?
   (a) $\begin{pmatrix} 20 \\ 10 \\ -60 \end{pmatrix}$
   (b) $[-12,-6,18]$
   (c) $[0,0,0]$
   (d) $\begin{pmatrix} 0.4 \\ 0.2 \\ -0.6 \end{pmatrix}$
   (e) $\begin{pmatrix} \sqrt{32} \\ \sqrt{8} \\ -\sqrt{72} \end{pmatrix}$
   (f) $[8,4,-10]$

9. Draw, on one picture, the vectors $7\hat{\jmath}$ and $4\hat{\imath} - 4\hat{\jmath}$ as arrows starting at $(0,0)$. What is the angle between these two vectors?

10. Let P be the point $(5,2)$ and let Q be the point $(1,9)$. Describe the vector $\left[\begin{smallmatrix} 5 \\ 2 \end{smallmatrix}\right] - \left[\begin{smallmatrix} 1 \\ 9 \end{smallmatrix}\right]$ in words, *without doing any calculations.*

11. Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be as in the image below.



    Write a true equation of the form __ + __ = __ using these vectors.

12. Write two true equations of the form __ − __ = __ using vectors from Task 11.

13. Calculate the following vectors or scalars:
    (a) $|4\hat{\imath} - 4\hat{\jmath}|$
    (b) $\big|[0,7]\big|$
    (c) $\dfrac{[3,2]}{\big|[3,2]\big|}$
    (d) $\left|\dfrac{[3,2]}{\big|[3,2]\big|}\right|$

14. Calculate each of the following. Each answer will be either a scalar *formula* or a vector *formula* involving t.
    (a) $5\begin{pmatrix} 3 \\ 2 \end{pmatrix} + t\begin{pmatrix} 7 \\ 1 \end{pmatrix}$
    (b) $t + \left|\left[\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right]\right|$
    (c) $\big|t[3,2]\big|$
    (d) $\big|[1+t, 1-t]\big|^2$

15. A parallelogram has the vector $\vec{a} = [5, 2]$ along one edge and $\vec{b} = [3, 8]$ along another edge. Compute the lengths of the two diagonals of the parallelogram.

16. Give a vector that is parallel to $\vec{v} = [8, -1, 4]$ but has length 1.

The **dot product** (also called **scalar product**) of $\vec{u}$ and $\vec{v}$ is written $\vec{u} \cdot \vec{v}$ and can be calculated as either

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

or

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos(\text{angle between } \vec{u} \text{ and } \vec{v}).$$

Two vectors are called **orthogonal** if their dot product is 0.

17. Calculate $\begin{pmatrix} 5 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} -8 \\ 1 \end{pmatrix}$.

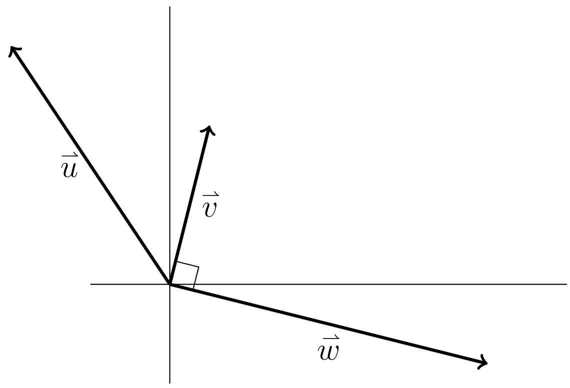
18. Calculate $(4\hat{\imath} - 4\hat{\jmath}) \cdot (7\hat{\jmath})$ in two ways:

    (a) by using $(4)(0) + (-4)(7)$.

    (b) by using $(4\sqrt{2})(7)\cos(135°)$. (See Tasks 13(a), 13(b), and 9.)

19. Find the angle between $\begin{pmatrix} -3 \\ \sqrt{3} \end{pmatrix}$ and $\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$.

20. Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be as in the image below.



    (a) Is $\vec{u} \cdot \vec{v}$ positive, negative, or zero?

    (b) Is $\vec{u} \cdot \vec{w}$ positive, negative, or zero?

    (c) Is $\vec{v} \cdot \vec{w}$ positive, negative, or zero?

21. For each formula below, state whether it represents a scalar (number), a vector, or "nonsense" (meaning it is not a legal operation; for example, $\vec{v}+5$ is nonsense.

    (a) $\vec{a} + \vec{b}$
    (b) $\vec{u} \cdot \vec{v}$
    (c) $\vec{a}\vec{b}$
    (d) $t\vec{a}$
    (e) $t + \vec{v}$
    (f) $(t + s)\vec{u}$
    (g) $\vec{n}/s$
    (h) $\vec{a} - s$
    (i) $\vec{c} + s\vec{b}$
    (j) $t(\vec{a} + \vec{b}) - \vec{c}$
    (k) $(\vec{a} \cdot \vec{b})\vec{c}$
    (l) $\vec{0} - \vec{a}$
    (m) $\vec{0} \cdot \vec{w}$
    (n) $\begin{pmatrix} 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 8 \end{pmatrix}$
    (o) $[4, 2] \cdot [s, t]$
    (p) $\vec{w} \cdot [s, t]$
    (q) $|\vec{u}|$

(r) $\left|\left[9, 2, \frac{1}{2}\right]\right|$ (t) $|\vec{a}| + (\vec{b} \cdot \vec{c})$ (v) $(\vec{a})^2$

(s) $|\vec{w}|\vec{v}$ (u) $|\vec{a}|(\vec{b} \cdot \vec{c})$ (w) $|\vec{a}|^2$

22. State whether each pair of vectors below is parallel, perpendicular, or neither.

(a) $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 7 \end{pmatrix}$

(b) $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0.4 \end{pmatrix}$

(c) $2\hat{\imath} - 8\hat{\jmath}$ and $-8\hat{\imath} + 2\hat{\jmath}$

(d) $\begin{pmatrix} 10 \\ -6 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$

(e) $9\hat{\imath} + 11\hat{\jmath} - 29\hat{k}$ and $2\hat{\imath} + j + \hat{k}$

(f) $32\hat{\imath} + 180\hat{\jmath}$ and $32\hat{\imath} + 7\hat{k}$

23. Give an example of a vector that is perpendicular to $\vec{v} = \hat{\imath} + 9\hat{\jmath} + 4\hat{k}$.

24. Knowing that

$$\cos(19.5°) \approx \tfrac{33}{35}, \quad \cos(25.2°) \approx \tfrac{19}{21}, \quad \cos(31°) \approx \tfrac{6}{7}, \quad \cos(62.96°) \approx \tfrac{15}{33},$$

find the acute angle between $[6, 3, 6]$ and $[6, 9, 18]$.

A **linear combination** of a collection of vectors is a sum (+) of scalar multiples of those vectors. (A linear combination of one vector is simply a scalar multiple of that one vector.)

25. Write $\begin{pmatrix} 13 \\ 3 \end{pmatrix}$ as a linear combination of $\hat{\imath} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\hat{\jmath} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

26. Write $\begin{pmatrix} 13 \\ 3 \end{pmatrix}$ as a linear combination of $\vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

27. Why is it impossible to write $\begin{pmatrix} 13 \\ 3 \end{pmatrix}$ as a linear combination of $\vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{c} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$?

28. For the vectors

$$\vec{v_1} = 2\hat{\imath} + 9\hat{\jmath} - 6\hat{k}, \qquad \vec{v_2} = 4\hat{\imath} + 2\hat{\jmath} - 6\hat{k}, \qquad \vec{v_3} = -8\hat{\jmath} + 3\hat{k},$$

either write $\vec{v_1}$ as a a linear combination of the other vectors or explain why it is not possible to do so.