Linear Algebra, Winter 2022

List 2

Lines and planes

29. Calculate the distance between the points $(5, 1, 1)$ and $(-8, 0, 7)$.
    $\sqrt{234}$, or $\boxed{3\sqrt{26}}$

> The **cross product** of $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$ is written $\vec{a} \times \vec{b}$. It is the vector that is perpendicular to $\vec{a}$ and $\vec{b}$, has length $|\vec{a}||\vec{b}|\sin(\theta)$, where θ is the angle between $\vec{a}$ and $\vec{b}$, and whose direction obeys the Right-Hand Rule.
>
> To calculate $\vec{a} \times \vec{b}$, we can use $\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\hat{\jmath} \times \hat{k} = \hat{\imath}$, $\hat{k} \times \hat{\imath} = \hat{\jmath}$ and very careful algebra, or use the direct formula
>
> $$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\hat{\imath} + (a_3b_1 - a_1b_3)\hat{\jmath} + (a_1b_2 - a_2b_1)\hat{k}.$$

30. (a) Give an example of a vector parallel to $9\hat{\imath} + 6\hat{\jmath} + 2\hat{k}$ that has magnitude 2. How many vectors with those two properties exist?

    There are only $\boxed{\text{two}}$ examples: $\boxed{\left[\frac{18}{11}, \frac{12}{11}, \frac{4}{11}\right] \text{ and } \left[\frac{-18}{11}, \frac{-12}{11}, \frac{-4}{11}\right]}$.

    (b) Give an example of a vector perpendicular to $9\hat{\imath} + 6\hat{\jmath} + 2\hat{k}$ that has magnitude 2. How many vectors with those two properties exist? There are infinitely many examples. One is $\left[\frac{2}{\sqrt{85}}, 0, \frac{-9}{\sqrt{85}}\right]$.

    (c) Give an example of a vector that is perpendicular to $9\hat{\imath} + 6\hat{\jmath} + 2\hat{k}$ and also perpendicular to $5\hat{\imath} + 5\hat{k}$ and has magnitude 2. How many vectors with those three properties exist? There are exactly two: $\boxed{\pm\left[\frac{12}{11}, -\frac{14}{11}, -\frac{12}{11}\right]}$.

> The vector $\vec{r}$ is used for $[x, y]$ in 2D and $[x, y, z]$ in 3D.

31. Re-write $\begin{cases} x = 2 + 5t \\ y = 3t \\ z = 9 \end{cases}$ as a single equation using vectors.

    There are multiple correct answers, including $\vec{r} = \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix} + t \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}$ and $\boxed{\vec{r} = \begin{bmatrix} 2 + 5t \\ 3t \\ 9 \end{bmatrix}}$.

32. Describe in words the set of points (x, y) that satisfy
    (a) $y = x^2$. $\boxed{\text{parabola}}$
    (b) $x^2 + y^2 = 100$. $\boxed{\text{circle}}$
    (c) $|x\hat{\imath} + y\hat{\jmath}| = 100$. $\boxed{\text{circle}}$ This has exactly the same points as (b).
    (d) $|\vec{r}| = 100$. $\boxed{\text{circle}}$ This has exactly the same points as (b).
    (e) $3x + 2y = 0$. $\boxed{\text{line}}$[1]

[1] The task explicitly states that this is about sets in $\mathbb{R}^2$. The set of points (x, y, z) that satisfy $3x + 2y = 0$ is a plane in $\mathbb{R}^3$.

(f) $\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0$. $\boxed{\text{line}}$ This has exactly the same points as (e).

(g) $(3\hat{\imath} + 2\hat{\jmath}) \cdot \vec{r} = 0$. $\boxed{\text{line}}$ This has exactly the same points as (e).

A vector that is parallel to a line is called a **direction vector** for the line.
The line through (x_0, y_0, z_0) with direction $\vec{d} = [a, b, c]$ has "vector equation"

$$\vec{r} = \vec{p} + t\,\vec{d}.$$

To describe it without vectors we can use the *three* "parametric equations"

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

33. Describe the line through the point (2, 2.4, 3.5) and parallel to the vector $3\,\hat{\imath} + 2\,\hat{\jmath} - \hat{k}$ using the parameter t in

    (a) one vector equation. $\boxed{\vec{r} = [2, 2.4, 3.5] + t[3, 2, -1]}$

    (b) three scalar equations. $\boxed{x = 2 + 3t, \quad y = 2.4 + 2t, \quad z = 3.5 - t}$

34. Write an equation for the line through points $(2, 3, 1)$ and $(-3, 7, 0)$.

    $\vec{d} = [2, 3, 1] - [-3, 7, 0] = [5, -4, 1]$
    An equation (really, 3 equations) for this line is $\boxed{x = 2 + 5t,\ y = 3 - 4t,\ z = 1 + t}$.
    Other correct answers are also possible such as $\boxed{x = -3 + 5t,\ y = 7 - 4t,\ z = t}$.

35. Determine whether the point $(9, -10, 3)$ is on the line

$$x = 5 + 2t, \qquad y = 2 - 6t, \qquad z = -1 - t.$$

    This is the same as asking whether there is a single value of t for which

$$9 = 5 + 2t \quad \text{and} \quad -10 = 2 - 6t \quad \text{and} \quad 3 = -1 - t.$$

    The point $(9, -10, 3)$ is not on the line.

36. Is the point $(4, 8, 7)$ on the line $\vec{r} = [1, 2, 6] + [3, 8, 9]t$?

    The point $(4, 8, 7)$ is not on the line.

37. Find the point where the lines

$$\begin{aligned} L_1 : &\quad x = 35 + 2t,\ y = 9 + t,\ z = -24 - 4t \\ L_2 : &\quad x = -3 + 2s,\ y = 16 - 3s,\ z = 13 + 2s \end{aligned}$$

    intersect. That is, find one value of t and one value of s such that

$$35 + 2t = -3 + 2s \quad \text{and} \quad 9 + t = 16 - 3s \quad \text{and} \quad -24 - 4t = 13 + 2s$$

    and then either use t to calculate the (x, y, z)-coordinates of the point from L_1 or use s to calculate the (x, y, z)-coordinates of the point from L_2.

    $t = -25/2$ and $s = 13/2$ both give the point $\boxed{(10, \frac{-7}{2}, 26)}$

38. Do the lines

$$\begin{aligned} L_1: &\quad x = 3+t,\ y = 2-2t,\ z = 1+5t \\ L_2: &\quad x = 8-3s,\ y = 1+s,\ z = -10+5s \end{aligned}$$

intersect? If so, find the intersection point and the angle between the two lines at that point (using a calculator for $\cos^{-1}$).

The lines intersect at the point $\boxed{(13/5,\ 14/5, -1)}$ (this corresponds to $t = -2/5$ for L_1 and $s = 9/5$ for L_2). The angle between the lines is the angle between the direction vectors $\vec{a} = [1, -2, 5]$ and $\vec{b} = [-3, 1, 5]$, which is

$$\arccos\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right) = \boxed{\arccos\left(\frac{20}{5\sqrt{42}}\right) \approx 0.9056 \approx 51.89^\circ.}$$

39. Do the lines

$$\begin{aligned} L_1: &\quad x = 3+t,\ y = 2-2t,\ z = 1+5t \\ L_3: &\quad x = 5+2s,\ y = -6-s,\ z = 7-4s \end{aligned}$$

intersect? If so, find the intersection point and the angle between the two lines at that point (using a calculator for $\cos^{-1}$).

$\boxed{\text{No intersection}}$

40. Explain why the parametric equations

$$\begin{cases} x = 1-t \\ y = 2-3t \\ z = 4t \end{cases} \quad \text{and} \quad \begin{cases} x = 2s \\ y = -1+6s \\ z = 4-8s \end{cases}$$

describe the same line.

The direction vectors $[-1, -3, 4]$ and $[2, 6, -8]$ are parallel, so the lines are in the same direction, AND the point $(1, 2, 0)$ from $t = 0$ on the first line is also on the second line (when $s = \frac{1}{2}$).

A vector that is perpendicular to a plane is called a **normal vector** for the plane.

The plane through point (x_0, y_0, z_0) with normal vector $\vec{n} = [a, b, c]$ has

"vector equation" $\vec{n}\cdot(\vec{r}-\vec{p}) = 0$

"scalar equation" $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

"standard equation" $ax + by + cz = d$

where $\vec{p} = [x_0, y_0, z_0]$ and $d = \vec{n}\cdot\vec{p}$.

41. Find a scalar equation for the plane through the origin perpendicular to the vector $[1, -2, 5]$. With vectors, $[1, -2, 5]\cdot[x, y, z] = 0$, so this is $\boxed{x - 2y + 5 = 0}$.

42. Find an equation for the plane through the point $(1, -1, -1)$ parallel to the plane $5x - y - z = 6$. The first plane is normal to $\vec{n} = [5, -1, -1]$, so this one is also normal to the same vector. One equation for the plane is

$$\boxed{[5, -1, -1]\cdot([x, y, z] - [1, -1, -1]) = 0.}$$

By expanding the dot product, we get the equivalent equation

$$\boxed{5(x-1)+(-1)(y+1)+(-1)(z+1)=0.}$$

This can be simplified to just

$$\boxed{5x-y-z=7.}$$

43. Find the intersection point of the line L and the plane P, where

$$\begin{aligned} L: &\qquad x=t,\ y=1-2t,\ z=-3+2t \\ P: &\qquad 3x-y-2z=3. \end{aligned}$$

The value of t for this intersection point satisfies

$$\begin{aligned} 3x-y-2z &= 3 \\ 3(t)-(1-2t)-2(-3+2t) &= 3 \\ t+5 &= 3 \\ t &= -2 \end{aligned}$$

The coordinates of this point are therefore

$$x=-2, \quad y=1-2(-2)=5, \quad z=-3+2(-2)=-7,$$

so the point is $\boxed{(-2,5,-7)}$.

44. Find the distance between the point $(-6,3,5)$ and the plane $x-2y-4z=8$ using the following steps:

(a) Give a vector that is perpendicular to the plane $x-2y-4z=8$.

(b) Give an equation for the line through $(-6,3,5)$ perpendicular to the plane $x-2y-4z=8$ (that is, parallel to the vector from part (a)).

(c) Find the point where the plane $x-2y-4z=8$ and the line from part (b) intersect.

(d) Calculate the distance between $(-6,3,5)$ and the point from part (c). This is exactly the distance between $(-6,3,5)$ and the plane $x-2y-4z=8$.

The closest point to P on the plane $x-2y-4z=8$ is on the line through P that is perpendicular to $x-2y-4z=8$. That line has direction vector $\vec{v}=[1,-2,-4]$, the same as the normal vector for the plane, so that line is

$$\vec{r}=[-6,3,5]+[1,-2,-4]t$$

or

$$\vec{r}=[-6+t,3-2t,5-4t].$$

The intersection of this line with $x-2y-4z=8$ occurs when

$$\begin{aligned} (-6+t)-2(3-2t)-4(5-4t) &= 8 \\ -32+21t &= 8 \\ t &= 40/21 \end{aligned}$$

which corresponds to the point

$$\left[-6+\tfrac{40}{21}, 3-2\tfrac{40}{21}, 5-4\tfrac{40}{21}\right] = \left[\tfrac{-86}{21}, \tfrac{-17}{21}, \tfrac{-55}{21}\right].$$

The distance between $(-6, 3, 5)$ and $(\frac{-86}{21}, \frac{-17}{21}, \frac{-55}{21})$ is

$$\sqrt{(-6-\tfrac{-86}{21})^2 + (3-\tfrac{-17}{21})^2 + (5-\tfrac{-55}{21})^2} = \boxed{\frac{40}{\sqrt{21}}}.$$

45. Consider the planes

$$\begin{aligned} P_1 : &\quad 8(x-1) + 6(y+3) + 16(z+7) = 0, \\ P_2 : &\quad 4x + 3y + 8z = 27. \end{aligned}$$

(a) Do the planes intersect? $\boxed{\text{No intersection}}$ because the normal vectors $\vec{n}_1 = [8, 6, 16]$ and $\vec{n}_2 = [4, 3, 8]$ are parallel and the two planes are not the same plane (because, for example, $(0, 7, 0)$ is on P_2 but not P_1).

(b) If the planes intersect, find the angle between the planes.

(c) If the planes intersect, give an equation for the line that is their intersection.

46. Consider the planes

$$\begin{aligned} P_1 : &\quad 8(x-1) + 6(y+3) + 16(z+7) = 0, \\ P_2 : &\quad 2x - 3y + 7z = 1. \end{aligned}$$

(a) Do the planes intersect? $\boxed{\text{Yes}}$

(b) If the planes intersect, find the angle between the planes. This is the angle between the normal vectors $\vec{n}_1 = [8, 6, 16]$ and $\vec{n}_2 = [2, -3, 7]$, which is $\boxed{\cos^{-1}(\frac{55}{\sqrt{5518}}) \approx 42.2°}$.

(c) If the planes intersect, give an equation for the line that is their intersection. One possible description of the line is $\boxed{x = 5 + 15t, \quad y = -11 - 4t, \quad z = -6 - 6t}$.

47. Find parametric equations for the line that is the intersection of the two planes $x + y = 3$ and $y + z = 1$.

The point $(2, 1, 0)$ is on both planes, so it is part of their intersection. The line will be perpendicular to the normal vectors for each plane. That is, it will be perpendicular to $[1, 1, 0]$ and $[0, 1, 1]$. Such a vector $[a, b, c]$ can be found using the cross product

$$[1, 1, 0] \times [0, 0, 1] = [1, -1, 1]$$

or by finding any solution other than $[0, 0, 0]$ to the system

$$\begin{cases} 1a + 1b + 0c = 0 \\ 0a + 1b + 1c = 0 \end{cases} \quad \to \quad \begin{cases} a + b = 0 \\ b + c = 0 \end{cases}$$

(there are infinitely many such solutions, all scalar multiples of $[1, -1, 1]$).

The line through $(2, 1, 0)$ with direction vector $[1, -1, 1]$ is $\boxed{x = 2 + t,\ y = 1 - t,\ z = t}$.

48. Find the acute angle between the line and the plane (at the point where they intersect):

$$L: \qquad x = 5+\sqrt{3}t,\ y=\sqrt{5}+3t,\ z=-1+2t;$$
$$P: \qquad (x+6)+\sqrt{3}(y-4)+2\sqrt{3}\,z=0.$$

This is $90°$ (or $\frac{\pi}{2}$) minus the angle between $\vec{d}$ (line's direction vector) and $\vec{n}$ (plane's normal vector).

$$\vec{d} = [\sqrt{3},3,2] \qquad \left|\vec{d}\right| = \sqrt{3+9+4} = \sqrt{16} = 4$$
$$\vec{n} = [1,\sqrt{3},2\sqrt{3}] \qquad \left|\vec{n}\right| = \sqrt{1+3+4\cdot 3} = \sqrt{16} = 4$$
$$\vec{d}\cdot\vec{n} = \sqrt{3}+3\sqrt{3}+4\sqrt{3} = 8\sqrt{3}$$

Therefore $8\sqrt{3} = (4)(4)\cos\theta$, so $\cos\theta = \frac{8\sqrt{3}}{16} = \frac{\sqrt{3}}{2}$ and $\theta = 30°$.

The final answer is $90° - 30° = \boxed{60° \text{or} \frac{\pi}{3}}$. (If the task did not specify that the angle should be acute, then either $90°+30° = 120°$ or $90°-30° = 60°$ would be correct.)

49. If a is a scalar, $\vec{b}$ is a 2D vector, and $\vec{c}$ is a 3D vector, which of the following calculations are possible?

(a) $5+a$ possible

(b) $5+\vec{b}$ impossible

(c) $5a$ possible

(d) $5\vec{c}$ possible

(e) $\vec{b}+\vec{c}$ impossible

(f) $\vec{b}\cdot\vec{c}$ impossible

(g) $\dfrac{\vec{c}}{5}$ possible

(h) $\dfrac{\vec{c}}{|\vec{b}|}$ possible

(i) $\dfrac{\vec{c}}{\vec{b}}$ impossible

(j) $\dfrac{\vec{c}}{\vec{c}}$ impossible

(k) $\vec{a}^3$ impossible

(l) $5+|\vec{c}|^3$ possible

(m) $|5\vec{a}|^3$ possible

(n) $|\vec{a}|+|\vec{c}|$ possible

(o) $|\vec{c}|\vec{a}$ possible