Linear Algebra, Winter 2022

List 3

Matrices

A **matrix** is a grid of numbers. The **dimensions** of a matrix are written in the format "$m \times n$", spoken as "m by n", where m is the number of rows and n is the number of columns (write both numbers; do <u>not</u> multiply them).

50. Give the dimensions of the following matrices:

(a) $\begin{bmatrix} -92 & 8 \\ -78 & -67 \end{bmatrix}$

(b) $\begin{bmatrix} -36 \\ 72 \\ -12 \end{bmatrix}$

(c) $\begin{bmatrix} 75 & 89 & 50 \\ -5 & -81 & 34 \end{bmatrix}$

(d) $\begin{bmatrix} -13 & -63 & -5 \\ 0 & -66 & \frac{1}{2} \\ 31 & \frac{5}{22} & \frac{8}{11} \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 58 & -65 & 40 & 8 & -1 & 26 \\ -74 & 58 & -92 & -4 & -21 & 74 \end{bmatrix}$

51. Assume A and B are 3×3 matrices, and $\vec{u}$ and $\vec{v}$ are 3×1 column vectors. For each formula below, does it represent a scalar, a vector, a matrix, or nonsense?

(a) $A + B$

(b) AB

(c) $A + \vec{u}$

(d) $A\vec{u}$

(e) ${}^{A}/_{\vec{u}}$

(f) $\vec{v}B$

(g) ${}^{\vec{v}}/_{B}$

(h) $A + \vec{u}$

(i) $AB\vec{v}$

(j) $(A + B)(\vec{u} + \vec{v})$

(k) $A(\vec{u} \times \vec{v})$

(l) $(\vec{u} \times \vec{v})A$

How to multiply matrices: The number in row i and column j of the matrix AB is the dot product of (Row i from matrix A) and (Column j from matrix B).

52. If A is a 2×2 matrix, B is a 3×3 matrix, and C is a 3×2 matrix, which of the following exist?

(a) AA

(b) AB

(c) AC

(d) BA

(e) BB

(f) BC

(g) CA

(h) CB

(i) CC

(j) ABC

(k) BCA

(ℓ) ACA

53. (a) Calculate $\begin{bmatrix} 1 & 2 \\ 5 & -8 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & 12 \end{bmatrix}$.

(b) Calculate $\begin{bmatrix} 3 & 0 \\ 2 & 12 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 5 & -8 \end{bmatrix}$.

(c) Compare your answers to parts (a) and (b).

54. Compute the following:

(a) $\begin{bmatrix} 1 & 14 & 8 \\ 6 & -1 & 14 \\ 5 & 11 & -3 \end{bmatrix} + \begin{bmatrix} 11 & 2 & 10 \\ -2 & 14 & 8 \\ -2 & 3 & -4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 14 & 8 \\ 6 & -1 & 14 \\ 5 & 11 & -3 \end{bmatrix} - \begin{bmatrix} 11 & 2 & 10 \\ -2 & 14 & 8 \\ -2 & 3 & -4 \end{bmatrix}$

(c) $3\begin{bmatrix} 0 & -4 & 0 \\ -1 & -1 & 3 \\ -2 & 5 & 14 \end{bmatrix}$

(d) $\frac{1}{6}\begin{bmatrix} 9 & 14 \\ 6 & 10 \end{bmatrix}$

(e) $\begin{bmatrix} 8 & 5 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix}$

(f) $\begin{bmatrix} 9 & -2 \\ 8 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix}$

(g) $\begin{bmatrix} -5 & 5 & 7 \\ -2 & -3 & 2 \\ -2 & 1 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 9 \\ 8 \end{bmatrix}$

(h) $\begin{bmatrix} 4 & 8 & 0 \\ -3 & 3 & -3 \\ 8 & 5 & -2 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \\ 7 \end{bmatrix}$

55. Compute $\begin{bmatrix} 1 & -\sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$.

56. Compute the following, if they exist:

(a) $\begin{bmatrix} 9 & -4 \\ -5 & -5 \end{bmatrix}\begin{bmatrix} 8 & 1 \\ 0 & -3 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 5 & 22 \\ 8 & -13 & 4 \end{bmatrix}\begin{bmatrix} 19 & 0 & 35 & 6 \\ 0 & 2 & 2 & 6 \\ 9 & 1 & 19 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 19 & 0 & 35 & 6 \\ 0 & 2 & 2 & 6 \\ 9 & 1 & 19 & -1 \end{bmatrix}\begin{bmatrix} 4 & 5 & 22 \\ 8 & -13 & 4 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 0 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 0 & 2 & -1 & 2 & 7 \\ 3 & -4 & -1 & 1 & 8 \end{bmatrix}$

(e) $\begin{bmatrix} -2 & -4 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 7 & 8 \\ 2 & 8 \end{bmatrix}$

(f) $\begin{bmatrix} -4 & -3 & -5 \\ 24 & 6 & 29 \end{bmatrix}\begin{bmatrix} 4 & 13 & 0 \\ 2 & -26 & 9 \end{bmatrix}$

57. Compute the following:

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 8 & 2 \\ 3 & -3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 14 & 21 \\ -11 & 23 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 99 & \frac{1}{10} \\ -37 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -4 & -2 & 1 \\ 5 & 2 & 5 \\ 7 & 4 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 15 & 11 & -15 \\ 17 & 10 & -8 \\ 0 & 0 & -13 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 59 & 28 & 58 \\ 61 & 44 & 67 \\ 22 & 39 & 17 \end{bmatrix}$

(g) $\begin{bmatrix} 59 & 28 & 58 \\ 61 & 44 & 67 \\ 22 & 39 & 17 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

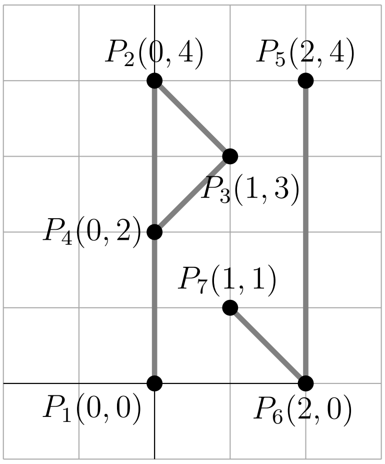
(h) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -39 & -66 & 84 & 66 & -10 \\ -47 & -5 & 17 & -59 & -3 \\ -94 & -90 & -5 & 86 & 31 \\ 25 & 80 & 0 & 35 & 19 \\ -72 & 40 & 99 & 48 & 57 \end{bmatrix}$

58. For each of the points P_1 through P_7, calculate

$$P_i{}' = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} P_i.$$

(For example, for $P_5{}' = \left[\begin{smallmatrix} 1 & 1/2 \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 4 \\ 4 \end{smallmatrix}\right]$.) Plot the points $P_1{}', \ldots, P_7{}'$ on a new grid. Connect $P_1{}' \to P_2{}' \to P_3{}' \to P_4{}'$ with line segments, and connect $P_5{}' \to P_6{}' \to P_7{}'$.

Congratulations. You can write in italics!



59. If $\begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix} M = \begin{bmatrix} 8 & 25 & 12 \\ 14 & 45 & 22 \end{bmatrix}$, what are the dimensions of matrix M?

60. Give the dimensions of the matrix $\begin{bmatrix} 2 & -8 \\ 1 & 5 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 & 11 & 4 \\ -2 & -8 & 6 & 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 0 \\ 1 \\ -9 \end{bmatrix} \begin{bmatrix} \frac{2}{7} & 1 & \frac{4}{7} \end{bmatrix}$.

(Do *not* compute the matrix product.)

61. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 5 & 2 \end{bmatrix}$, and $E = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 1 \end{bmatrix}$. Write all the products of two matrices from this list that exist (e.g., AA exists, but AC does not).

Lecture from 6 October covers only to this point.

A function f with vector inputs and outputs is a **linear transformation** if

$$f(a\vec{u} + b\vec{v}) = af(\vec{u}) + bf(\vec{v})$$

for all scalars a, b and vectors $\vec{u}, \vec{v}$. Equivalently, $f(a\vec{u} + \vec{v}) = af(\vec{u}) + f(\vec{v})$ is enough, or $f(\vec{u} + \vec{v}) = f(\vec{u}) + f(\vec{v})$ and $f(a\vec{u}) = af(\vec{u})$ together.

62. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (x + y, x + 1)$. Show that f is not linear.

63. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (xy, 1)$. Show that T is not linear.

64. Which of the following are linear transformations?

(a) $g(x, y) = (y, x)$ (b) $L(x, y) = (0, y - 6x)$ (c) $K(x, y) = (6, y - x)$

65. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that $L(1, 1) = (3, -9)$ and $L(2, 0) = (6, 2)$. Calculate $L(13, 3)$.

66. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that $L(1, 0) = (4, -1)$ and $L(0, 1) = (-3, 2)$.

(a) Find $L(2, -4)$.

(b) Give a formula for $L(x, y)$ that works for any x and y.

67. Does there exist a linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$f(0,1) = f(4,3), \quad f(3,0) = f(-3,6), \quad f(1,3) = f(11,11) \ ?$$

68. Does there exist a linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$f(0,1) = f(4,3), \quad f(0,2) = f(8,6), \quad f(0,3) = f(10,8) \ ?$$

69. If $g : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation satisfying

$$g(2,0,0) = (8,8,8), \quad g(0,12,0) = (1,4,-6), \quad g(0,0,5) = (10,0,15),$$

calculate $g(-1,0,6)$.

If a linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $f(0,1) = (u_1,u_2)$ and $f(0,1) = (w_1,w_2)$, then

$$f(x,y) = \begin{bmatrix} u_1 & w_1 \\ u_2 & w_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

for any (x,y). We call $\left[\begin{smallmatrix} u_1 & w_1 \\ u_2 & w_2 \end{smallmatrix}\right]$ the "matrix for f" and write M_f for this matrix.

70. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $L(x,y) = (x+y,x)$. Show that L is linear and find the matrix for T.

71. The map $T_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T_\alpha\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

describes counter-clockwise rotation around the origin by an angle α.
Compute $T_{\pi/4}\left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right)$ and $T_\pi\left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right)$.

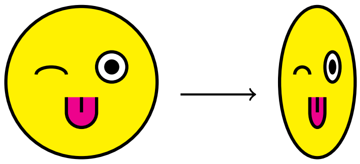
72. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation with

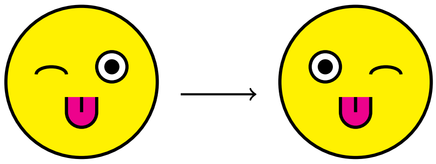
$$f(1,0,0) = (5,-1,4); \qquad f(0,1,0) = (2,1,-7); \qquad f(0,0,1) = (3,2,4).$$

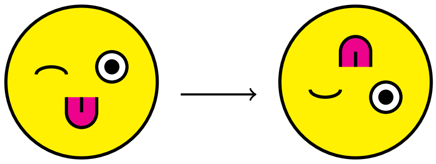
(a) Find $f(2,2,5)$.

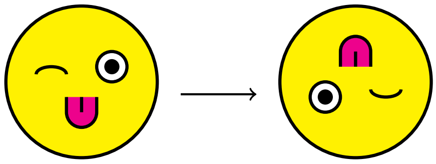
(b) Give the 3×3 matrix for f.

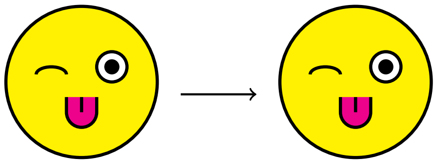
☆73. Match the following linear transformations with their matrices. (That is, which matrix describes (a)? Which matrix describes (b)? And so on.)

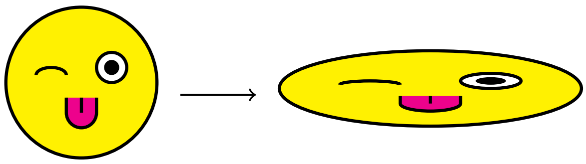
(a) 

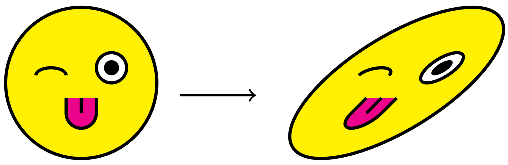
(b) 

(c) 

(d) 

(e) 

(f) 

(g) 

Matrices:

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad M_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad M_3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad M_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$M_5 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad M_6 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \qquad M_7 = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$$