Linear Algebra, Winter 2022

List 8

Polynomials

A **polynomial in x** is a function that can be written in the form

$$\_\_x^n + \_\_x^{n-1} + \cdots + \_\_x^2 + \_\_x + \_\_,$$

where each blank—called a **coefficient**—is a real or complex number, possibly including zero. A **real polynomial** is one whose coefficients are real numbers. In general, variables other than x can also be used (when complex numbers are involved, it is common, but *not* required, to use the variable z).

The **degree** of $f(x)$ is the highest power of x that has a non-zero coefficient.

187. Which of the following are polynomials (in any variable)?
   (a) $8x^2 + 4x + 1$ polynomial
   (b) $8z^2 + 4z + 1$ polynomial
   (c) $x^{10} + 5x^6 - 100x$ polynomial
   (d) $(z^5 - 2z + 1)(z + 1)$ polynomial
   (e) $(z^5 - 2z + 1)\sin(z)$ not polynomial
   (f) $3x^2 + 3x^{1/2} - 4$ not polynomial
   (g) $x^2 + 2^x$ not polynomial
   (h) $\sqrt{x^4 + 2x^2 + 1}$ polynomial $(x^2 + 1)$
   (i) $z + \overline{z}$ not polynomial

188. Which of the following are real polynomials (in any variable)?
   (a) $8x^2 + 4x + 1$ yes
   (b) $8z^2 + 4z + 1$ yes
   (c) $z^2 + 1$ yes
   (d) $z^2 + i$ yes
   (e) $(2 + i)x + (4 - i)$ no
   (f) $(z + i)(z - i)$ yes because this is $z^2 + 1$.

189. For each of the following, give the degree if the expression is a polynomial in x, and otherwise write "not a polynomial".
   (a) $\frac{5}{2}x^3 - 7x + 8$ degree 3
   (b) $9x^{10}$ degree 10
   (c) $6x^5 + \frac{1}{3}x + 5x^{-2}$ not a polynomial
   (d) $3x^2 + \sin(x)$ not a polynomial
   (e) $(x^2 + 2x - 1)^3$ degree 6
   (f) $5x$ degree 1
   (g) 5 degree 0

☆(h) 0 Some people say $f(x) = 0$ does not have a degree. Some people say its degree is $\boxed{-\infty}$.

(i) $\dfrac{8x^4+7}{2x}$ $\boxed{\text{not a polynomial}}$

(j) $\dfrac{8x^4+7x}{2x}$ $\boxed{\text{degree 3}}$

The number c is a **zero** (also called a **root**) of the polynomial $f(x)$ if $f(c) = 0$.

190. Find all the zeroes of $2x^2 + x - 15$. $\boxed{\frac{5}{2},\ -3}$

191. A cannonball fired at 400 m/s at an angle of 52° will have an initial vertical velocity of $400\sin(52°) \approx 315.2$ m/s, and it will have a height of

$$h(t) = \tfrac{-9.8}{2}t^2 + 315.2\,t$$

meters after t seconds. How many seconds will it take for the cannonball to reach the ground?

Without a calculator, $\boxed{\dfrac{2\times 315.2}{9.8}}$ is good enough. With calculator, $\boxed{64.3265}$.

192. Find all the roots of $x^5 - 6x^4 + 34x^3$. $\boxed{0, 3+5i, 3-5i}$

193. Given that 4 is one zero of $z^3 - 4z^2 + 49z - 196$, find all its roots. $\boxed{4, 7i, -7i}$

194. Given that $1+2i$ is a zero of $z^4 - 4z^3 + 12z^2 - 16z + 15$, find all its zeros.

$1 - 2i$ must also be a root.

$\dfrac{z^4 - 4z^3 + 12z^2 - 16z + 15}{(z-(1+2i))(z-(1-2i))}$ must be a polynomial.

$\dfrac{z^4 - 4z^3 + 12z^2 - 16z + 15}{z^2 - 2z + 5}$ must be a polynomial.

Long division:

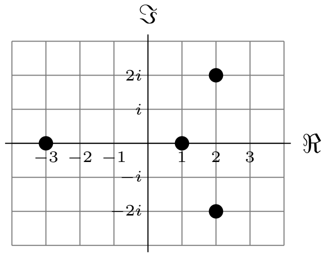
$$\begin{array}{r|lllll}
 & z^2 & -2z & +3 & & \\ \hline
z^2-2z+5 & z^4 & -4z^3 & +12z^2 & -16z & +15 \\
 & -(z^4 & -2z^3 & +5z^2) & & \\ \hline
 & & -2z^3 & +7z^2 & & \\
 & & -(-2z^3 & +4z^2 & -10z) & \\ \hline
 & & & 3z^2 & -6z & \\
 & & & -(3z^2 & -6z & +15) \\ \hline
 & & & & & 0
\end{array}$$

So we have $\dfrac{z^4 - 4z^3 + 12z^2 - 16z + 15}{z^2 - 2z + 5} = z^2 - 2z + 3$.

By the Quadratic Formula, the roots of $z^2 - 2z + 3$ are $1 \pm \sqrt{-2} = 1 \pm (\sqrt{2})i$.

The roots of the original polynomial are $\boxed{1+2i, 1-2i, 1+(\sqrt{2})i, 1-(\sqrt{2})i}$.

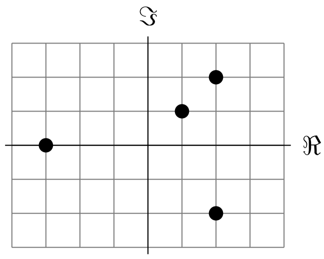
195. The figure below shows four points on the complex plane.



Give an example of a polynomial of degree 4 whose roots are exactly these four points.

$\boxed{(z+3)(z-(2+2i))(z-(2-2i))(z-1)}$ or $\boxed{z^4-2z^3-3z^2+28z-24}$ or any constant multiple of those.

196. The figure below shows four points on the complex plane.



(a) Does there exist a polynomial with real coefficients whose roots are exactly these four points? Why or why not? $\boxed{\text{No}}$ because if $1+i$ is a root of a real polynomial then $\overline{1+i} = 1-i$ must also be one of its roots.

(b) Does there exist a polynomial with complex coefficients whose roots are exactly these four points? Why or why not? $\boxed{\text{Yes}}$: $(z-c_1)\cdots(z-c_4)$ always works.

197. Does there exist a polynomial $f(x)$ with *integer* coefficients for which...

(a) $f(35) = 0$ and $f(7) = 0$? $\boxed{\text{yes}}$ Example: $f(x) = (x-35)(x-7) = x^2 - 42x + 245$

(b) $f(0) = 35$ and $f(7) = 0$? $\boxed{\text{yes}}$ Example: $f(x) = -5x + 35$

(c) $f(0) = 53$ and $f(7) = 0$? $\boxed{\text{no}}$

(d) $f(3i) = 5$ and $f(-3i) = 0$? $\boxed{\text{no}}$

198. Does there exist a polynomial $f(x)$ with *real* coefficients for which...
(a) $f(35) = 0$ and $f(7) = 0$? (b)-(d) each other condition from Task 197?

(a) $f(35) = 0$ and $f(7) = 0$? $\boxed{\text{yes}}$

(b) $f(0) = 35$ and $f(7) = 0$? $\boxed{\text{yes}}$

(c) $f(0) = 53$ and $f(7) = 0$? $\boxed{\text{yes}}$ Example: $f(x) = \frac{-53}{7}x + 53$

(d) $f(3i) = 5$ and $f(-3i) = 0$? $\boxed{\text{no}}$

199. Does there exist a polynomial $f(x)$ with *complex* coefficients for which...
(a) $f(35) = 0$ and $f(7) = 0$? (b)-(d) each other condition from Task 197?

(a) $f(35) = 0$ and $f(7) = 0$? $\boxed{\text{yes}}$

(b) $f(0) = 35$ and $f(7) = 0$? $\boxed{\text{yes}}$

(c) $f(0) = 53$ and $f(7) = 0$? $\boxed{\text{yes}}$

(d) $f(3i) = 5$ and $f(-3i) = 0$? $\boxed{\text{yes}}$ Example: $f(x) = \frac{-5i}{6}x + \frac{5}{2}$

200. Give the polynomial

$$f(x) = x^3 + \_\_x^2 + \_\_x + \_\_$$

for which $f(-1) = 0$, $f(3) = 0$, and $f(4) = 0$.
$(x+1)(x-3)(x-4) = \boxed{x^3 - 6x^2 + 5x + 12}$.

A polynomial is **reducible** if it can be factored into two non-constant polynomials; if not, it is **irreducible**. Whether a polynomial is reducible can depend on what kinds of numbers (e.g., real or complex) are allowed for the coefficients.

201. Are there polynomials $f(x)$ and $g(x)$ such that $f \cdot g = x^2 - 25$?
$\boxed{\text{Yes}}$: $(x-5) \cdot (x+5)$ Is $x^2 - 25$ reducible or irreducible? $\boxed{\text{reducible}}$

202. (a) Are there real polynomials f and g such that $f \cdot g = x^2 + 9$? $\boxed{\text{Yes}}$: $(1) \cdot (x^2 - 9)$

(b) Are there non-constant real polynomials f and g such that $f \cdot g = x^2 + 9$? $\boxed{\text{No}}$

(c) Are there non-constant polynomials f and g such that $f \cdot g = x^2 + 9$?
Hint: $(3i)^2 = -9$. $\boxed{\text{Yes}}$: $(x - 3i) \cdot (x + 3i)$

☆203. (a) Is 19 prime? $\boxed{\text{Yes}}$ Why or why not?

(b) Is 13 prime? Why or why not? If you can use "complex integers" (technically called "Gaussian integers") then $\boxed{\text{No}}$ because $13 = (3+2i)(3-2i)$. If you can only use standard integers, then $\boxed{\text{Yes}}$.

(c) Is 11 prime? $\boxed{\text{Yes}}$ Why or why not?

(d) Re-read Task 177 and then re-read 203(b).

204. For the polynomial

$$f(z) \quad = \quad z^4 + z^3 + z^2 + 3z - 6 \quad = \quad (z^2 + z - 2)(z^2 + 3),$$

(a) find all roots of $f(z)$. $\boxed{-2, 1, i\sqrt{3}, -i\sqrt{3}}$

(b) find all zeros of $f(z)$. (same as 204(a))

(c) solve $f(z) = 0$ for complex z. $\boxed{z = -2, z = 1, z = \sqrt{3}\,i, z = -\sqrt{3}\,i}$

(d) factor $f(z)$ into linear complex factors. $\boxed{(z+2)(z-1)(z-\sqrt{3}\,i)(z+\sqrt{3}\,i)}$

(e) factor $f(z)$ into irreducible complex factors. (same as 204(d))

205. Factor the following polynomials into irreducible real factors:

(a) x^3+x^2+x+1 $\boxed{(x+1)(x^2+1)}$

(b) x^3+x^2-x-1 $\boxed{(x+1)(x+1)(x-1) \text{ or } (x+1)^2(x-1)}$

(c) x^4-4x^3+8x $\boxed{x(x-2)(x^2-2x-4)}$

(d) x^4+5x^2+6 $\boxed{(x^2+2)(x^2+3)}$

206. Factor the following polynomials into irreducible complex factors:

(a) z^3+z^2+z+1 $\boxed{(z+1)(z+i)(z-i)}$

(b) z^3+z^2-z-1 $\boxed{(z+1)(z+1)(z-1) \text{ or } (z+1)^2(z-1)}$

(c) z^4-4z^3+8z $\boxed{z(z-2)(z-\sqrt{5}+1)(z+\sqrt{5}-1)}$

(d) z^4+5z^2+6 $\boxed{(z+i\sqrt{2})(z-i\sqrt{2})(z+i\sqrt{3})(z-i\sqrt{3})}$

207. Factor the polynomials

$$F(x) = x^3 - 6x^2 - 27x + 140$$
$$P(x) = x^3 - 14x^2 + 74x - 136$$

into irreducible real polynomials, knowing that $F(4) = P(4) = 0$.

$\boxed{F(x) = (x-4)(x-7)(x+5)}$ $\boxed{P(x) = (x-4)(x^2-10x+34)}$

208. Factor the polynomials from Task 207 into irreducible complex polynomials.

$\boxed{F(x) = (x-4)(x-7)(x+5)}$ $\boxed{P(x) = (x-4)(x-(5-3i))(x-(5+3i))}$

If r is root of the polynomial f, the **multiplicity** of r is the highest power m for which $(x-r)^m$ is a factor of f.

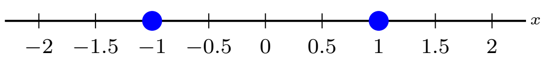
209. For $f(x) = (z-3)^4(z+2)$, what is the multiplicity of 3? $\boxed{4}$

210. For $g(z) = z^3 + 2z^2 - 7z + 4$, what is the multiplicity of 1?

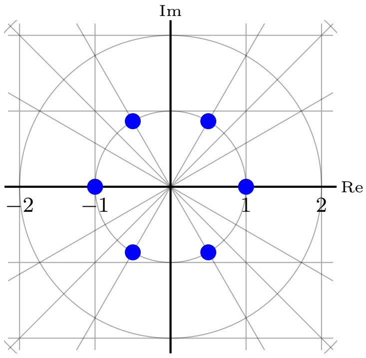
$g = (z-1)^2(z+4)$, so the multiplicity of 1 is $\boxed{2}$.

211. The only roots of $z^5 - 4z^4 + z^3 + 10z^2 - 4z - 8$ are -1 (with some multiplicity) and $+2$ (with some multiplicity). What is the sum of these multiplicities? $\boxed{5}$

212. (a) On a real number line (like the blank one shown below), put a dot at every point x for which $x^6 = 1$.



(b) On a complex plane (like the blank one shown below), put a dot at every point z for which $z^6 = 1$.



213. (a) Find all the roots of $f(z) = 1 + z + z^2$.

$\boxed{e^{120°i}, e^{-120°i}}$

In rectangular form, these are $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

(b) Find all the roots of $g(x) = 1 + x^2 + x^4$. (x can be complex)

$\boxed{e^{120°i}, e^{-120°i}, e^{60°i}, e^{-60°i}}$

In rectangular form, these are $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$, $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

(c) Find all the roots of $h(z) = 1 + z + z^2 + z^3 + z^4 + z^5$. Hint: $h(z) = \dfrac{1 - z^6}{1 - z}$.

$\boxed{e^{120°i}, e^{-120°i}, e^{60°i}, e^{-60°i}, e^{180°i}}$

In rectangular form, these are $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$, $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\frac{1}{2} - \frac{\sqrt{3}}{2}i$, -1.

☆214. Find the *sum* of all the roots (that is, add them together) of the polynomial

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9.$$

$\boxed{-1}$ In fact, the sum of the roots of $1 + z + z^2 + \cdots + z^n$ is exactly -1 for any whole number n.