Linear Algebra, Winter 2022

List 9

Review for Celebration of Knowledge 2

215. Find the eigenvalues of the matrix $M = \begin{bmatrix} -7 & 30 & 30 \\ 5 & 6 & 10 \\ -5 & 10 & 6 \end{bmatrix}$. $\boxed{-4, -7, 16}$

216. Find three non-parallel eigenvectors for the matrix from Task 215. For $\lambda = -4$ the eigenvector can be any non-zero scalar multiple of $\boxed{[0, 1, -1]}$. For $\lambda = -7$, $\boxed{3, -5, 5}$. For $\lambda = -16$, $\boxed{30, 19, 4}$.

217. Which of the following matrices are invertible?

(a) $\begin{bmatrix} -7 & 30 & 30 \\ 5 & 6 & 10 \\ -5 & 10 & 6 \end{bmatrix}$ $\boxed{\text{Yes}}$ (b) $\begin{bmatrix} 3 & -4 \\ -8 & 9 \end{bmatrix}$ $\boxed{\text{Yes}}$ (c) $\begin{bmatrix} 2 & -4 \\ -5 & 10 \end{bmatrix}$ $\boxed{\text{No}}$

☆218. Give an example of a matrix whose eigenvalues are 2 and 7. $\boxed{\begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}}$ is the simplest example. There are infinitely many others, such as $\begin{bmatrix} \frac{42}{11} & -\frac{70}{11} \\ -\frac{10}{11} & \frac{57}{11} \end{bmatrix}$.

219. The eigenvalues of a 6×6 real matrix include

$$2, \quad -4, \quad i, \quad 2+i.$$

What is the determinant of this matrix? $2(-4)(i)(-i)(2+i)(2-i) = \boxed{-40}$

220. For the complex numbers

$$z = 2\sqrt{3}e^{(\pi/3)i} = 2\sqrt{3}e^{60^\circ i} = \sqrt{3} + 3i \quad \text{and}$$
$$w = 2e^{(\pi/6)i} = 2e^{30^\circ i} = \sqrt{3} + i,$$

compute $\dfrac{z}{w}$, giving your answer in rectangular or exponential form (your choice).

$\dfrac{2\sqrt{3}e^{(\pi/3)i}}{2e^{(\pi/6)i}} = \dfrac{2\sqrt{3}}{2}e^{\left(\frac{\pi}{3}-\frac{\pi}{6}\right)i} = \boxed{\sqrt{3}e^{(\pi/6)i}}$ or $\boxed{\frac{3}{2} + \frac{\sqrt{3}}{2}i}$

221. For the numbers z and w from Task 220 compute $z - w$, giving your answer in rectangular or exponential form (your choice).

$\left(\sqrt{3} + 3i\right) - \left(\sqrt{3} + i\right) = \left(\sqrt{3} - \sqrt{3}\right) + (3i - i) = \boxed{2i}$ or $\boxed{2e^{(\pi/2)i}}$

222. For the number z and w from Task 220 compute $|\,\overline{z} \cdot w\,|$, giving your answer in rectangular or exponential form (your choice).

$|\overline{z} \cdot w| = |z| \cdot |w| = (2\sqrt{3}) \cdot 2 = \boxed{4\sqrt{3}}$

223. True or false?

(a) $\left|z + w\right| = \left|z\right| + \left|w\right|$ $\boxed{\text{false}}$

(b) $\left|zw\right| = \left|z\right| + \left|w\right|$ $\boxed{\text{false}}$

(c) $|zw| = |z| \cdot |w|$ $\boxed{\text{TRUE}}$

(d) $\arg(z + w) = \arg(z) + \arg(w)$ $\boxed{\text{false}}$

(e) $\arg(zw) = \arg(z)+\arg(w)$ $\boxed{\text{TRUE}}$ if z and w have arguments in $[-\pi/2, \pi/2]$ or if $\arg(u)$ is considered a "multivalued function". False if you require $\arg(u) \in (-\pi, \pi]$ for all $u \in \mathbb{C}$ (then, for example, $\arg(e^{135^\circ i} \cdot e^{135^\circ i}) = -90^\circ$). It is always true that

$$\arg(zw) = \arg(z) + \arg(w) + 2\pi n$$

for some integer n.

(f) $\arg(zw) = \arg(z) \cdot \arg(w)$ $\boxed{\text{false}}$

(g) $\overline{z + w} = \overline{z} + \overline{w}$ $\boxed{\text{TRUE}}$

(h) $\overline{zw} = \overline{z} + \overline{w}$ $\boxed{\text{false}}$

(i) $\overline{zw} = \overline{z} \cdot \overline{w}$ $\boxed{\text{TRUE}}$

(j) $\sqrt{z + w} = \sqrt{z} + \sqrt{w}$ $\boxed{\text{false}}$

(k) $\sqrt{zw} = \sqrt{z} + \sqrt{w}$ $\boxed{\text{false}}$

(ℓ) $\sqrt{zw} = \sqrt{z} \cdot \sqrt{w}$ $\boxed{\text{false}}$ if you use the "principle" square root. For example, $\sqrt{(-1)(-1)} = \sqrt{1} = 1$, but $\sqrt{-1} \cdot \sqrt{-1} = i \cdot i = -1$. True if square root is considered a "multivalued function". The property

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$

is true if at least one of a and b is a positive real number.

224. Convert the following numbers to exponential form, that is, $\_\_ e^{(\_\_ i)}$ where both blanks are real numbers and the first blank is non-negative.

(a) $4\cos(21^\circ) + 4\sin(21^\circ)i$ $\boxed{4e^{21^\circ i}}$

(b) $\cos(\pi/4) + \sin(\pi/4)i$ $\boxed{e^{(\pi/4)i}}$

(c) $\sqrt{2}\cos(\pi/4) + \sqrt{2}\sin(\pi/4)i$ $\boxed{\sqrt{2}e^{(\pi/4)i}}$

(d) $1 + i$ $\boxed{\sqrt{2}e^{(\pi/4)i}}$ Same number as (c).

(e) $4i$ $\boxed{4e^{(\pi/2)i}}$

(f) $\overline{-5 - 5i} = -5 + 5i = \boxed{5\sqrt{2}e^{(3\pi/4)i}}$

(g) $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ $\boxed{e^{(\pi/3)i}}$

(h) $\overline{6e^{5\pi/6}}$ $\boxed{6e^{-5\pi/6}}$

(i) $\frac{1}{2} - \frac{\sqrt{3}}{2}i$ $\boxed{e^{(-\pi/3)i}}$ or $e^{(5\pi/3)i}$

(j) $\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^6 = (e^{(\pi/4)i})^6 = \boxed{e^{(3\pi/2)i}}$

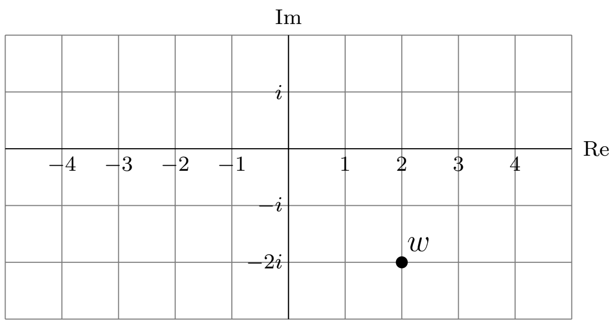
(k) $1 - \sqrt{3}\,i$ $\boxed{2e^{(-\pi/3)i}}$ or $2e^{(5\pi/3)i}$

(ℓ) $\sqrt{i}$ $\boxed{e^{(\pi/4)i}}$

(m) $\sqrt{3} - 3i$ $\boxed{2\sqrt{3}e^{(-\pi/3)i}}$ or $2\sqrt{3}e^{(5\pi/3)i}$

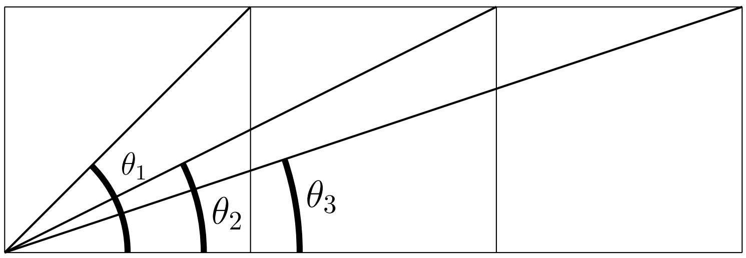
☆(n) $\sin(i)$ $\boxed{(\frac{e^2-1}{2e})e^{(\pi/2)i}}$ $\approx 1.175e^{(\pi/2)i}$

225. The figure below shows a complex number w.



Calculate $\overline{w^2}.\overline{(2-i)^2} = (2+i)^2 = \boxed{3+4i}$ or $5\,e^{53.13°i}$

☆226. The figure below shows three adjacent squares with line segments connecting some vertices. Use complex numbers to calculate the sum $\theta_1 + \theta_2 + \theta_3$ of the labeled angles.



$$\begin{aligned}\theta_1 + \theta_2 + \theta_3 &= \arg\big((1+i)(2+i)(3+i)\big)\\ &= \arg\big((1+3i)(3+i)\big)\\ &= \arg(10i)\\ &= \pi/2 \quad \text{or } 90°\end{aligned}$$

☆227. The vertices of a regular octagon lie on a circle of radius 1. What is the length of each side of the octagon?

$\boxed{\sqrt{2-\sqrt{2}}}$

Note: it is also possible to do this task using vectors or basic geometry. Often, 2D geometry tasks can be done using complex numbers. (For 3D, use vectors.)

228. One of the roots of

$$z^3 + 2z^2 - 38z + 80$$

is $3 - i$.

(a) Find all the complex roots of this polynomial. $\boxed{3 - i, \;\; 3 + i, \;\; -8}$

(b) Write $z^3 + 2z^2 - 38z + 80$ as a product of irreducible complex factors. $\boxed{(z + 8)(z - (3 - i))(z - (3 + i))}$

(c) Write $x^3 + 2x^2 - 38x + 80$ as a product of irreducible real factors. $\boxed{(x + 8)(x^2 - 6x + 10)}$

229. Find the real root(s) of $x^8 + x^6$ and the multiplicity of each root.

$\boxed{0 \text{ has multiplicity } 6.}$

230. Find the complex root(s) of $x^8 + x^6$ and the multiplicity of each root.

$\boxed{0 \text{ has multiplicity } 6. \; i \text{ has multiplicity } 1. \; -i \text{ has multiplicity } 1.}$

231. Give the cubic polynomial

$$f(x) = x^3 + \_\_x^2 + \_\_x + \_\_$$

for which $f(9) = 0$ and for which 13 is a zero with multiplicity 2.

$(x - 9)(x - 3)^2 = \boxed{x^3 - 35x^2 + 403x - 1521}$

232. One of the roots of

$$P(x) = 2x^5 - 5x^4 + 10x^2 - 10x + 3$$

is 1. What is its multiplicity?

$P(x) = (x - 1)^4(2x + 3)$, so 1 has $\boxed{\text{multiplicity } 4}$

233. The polynomial

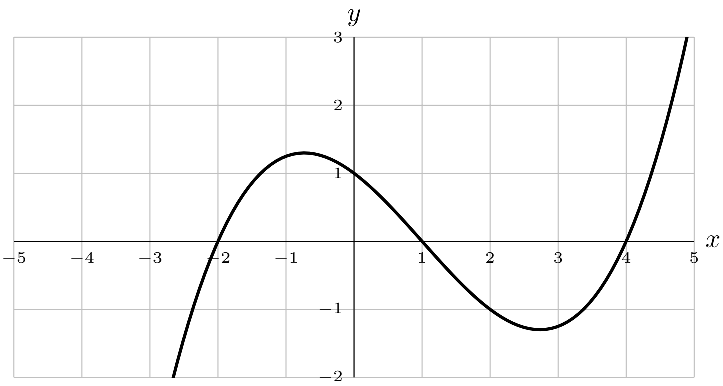
$$3z^6 - 25z^5 + 62z^4 - 132z^3 + 89z^2 - 107z + 30$$

has no repeated roots. How many linear complex polynomials are factors of this polynomial? $\boxed{6}$. (But it has only four irreducible *real* polynomial factors.)

☆234. Find all roots of $3z^6 - 25z^5 + 62z^4 - 132z^3 + 89z^2 - 107z + 30$.

$\boxed{\frac{1}{3}, 6, i, -i, 1 - 2i, 1 + 2i}$

235. Give the real polynomial of the form $\_\_x^3 + \_\_x^2 + \_\_x + \_\_$ whose graph is shown below.



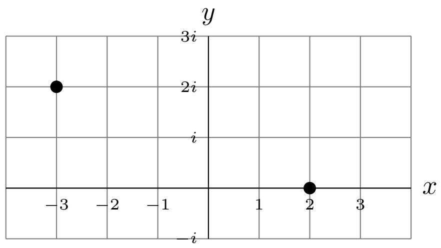
Roots at $x = -2$ and $x = 1$ and $x = 4$ mean we are looking for a multiple of

$$(x+2)(x-1)(x-4) \quad = \quad x^3 - 3x^2 - 6x + 8.$$

Since $f(0) = 1$ in the figure, we need to multiply the polynomial above by $\frac{1}{8}$. This gives

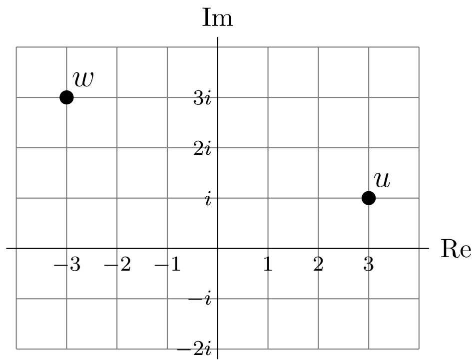
$$\boxed{\frac{1}{8}x^3 - \frac{3}{8}x^2 - \frac{3}{4}x + 1}$$

236. Give the real polynomial of the form $x^3 + \_\_x^2 + \_\_x + \_\_$ whose roots include the two complex numbers shown below.



$$\big(x - (-3+2i)\big)\big(x - (-3-2i)\big)\big(x - 2\big) = \boxed{x^3 + 4x^2 + x - 26}$$

237. Two complex numbers are shown in the figure below:



(a) Write the number u in rectangular form. $\boxed{3+i}$

(b) Write the number w in exponential form. $\boxed{3\sqrt{2}e^{135^\circ i}}$ or $\boxed{3\sqrt{2}e^{(3\pi/4)i}}$

(c) Calculate the complex conjugates $\overline{u}$ and $\overline{w}$. $\boxed{\overline{u} = 3 - i}$ $\boxed{\overline{w} = 3\sqrt{2}e^{-135^\circ i}}$

(d) Give an example of a polynomial with real coefficients whose only zeros are u and w, or state that such a polynomial does not exist. $\boxed{\text{doesn't exist}}$

(e) Give an example of a polynomial with complex coefficients whose only zeros are u and w, or state that such a polynomial does not exist. $\boxed{(z - (3+i))(z - (-3+3i))}$, which is also $\boxed{z^2 + (-4i)z + (6i - 12)}$

(f) Give an example of a degree-4 polynomial with real coefficients whose zeros include u and w (and possibly other points), or state that such a polynomial does not exist. $\boxed{z^4 - 8z^2 - 48z + 180}$