Math for Management, Winter 2023

List 1

Absolute value, inequalities, polynomials

1. Re-write the expressions below as either numbers or piecewise functions (do not use absolute value notation).

   (a) $|1-\sqrt{3}| = \boxed{\sqrt{3}-1}$

   (b) $|x+x^2| = \boxed{\begin{cases} x^2+x & \text{if } x \le 1 \text{ or } x \ge 0 \\ -x^2-x & \text{if } -1 < x < 0 \end{cases}}$

   (c) $x+|1-x|+2|x-2| = \boxed{\begin{cases} -4x+5 & \text{if } x < 0 \\ -2x+5 & \text{if } 0 \le x < 1 \\ 3 & \text{if } 1 \le x < 2 \\ 4x-5 & \text{if } x \ge 2 \end{cases}}$

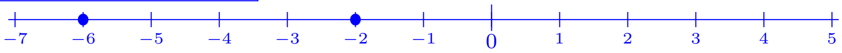
   (d) $|3x-8| = \boxed{\begin{cases} -3x+8 & \text{if } x < \frac{8}{3} \\ 3x-8 & \text{if } x \ge \frac{8}{3} \end{cases}}$

   (e) $|x+1|-x = \boxed{\begin{cases} -2x-1 & \text{if } x < -1 \\ 1 & \text{if } x \ge -1 \end{cases}}$

   (f) $|x-1|+\frac{x}{|x|}-|x+2| = \boxed{\begin{cases} 4 & \text{if } x < -2 \\ -2x & \text{if } -2 \le x \le 0 \\ -2x-1 & \text{if } 0 \le x < 1 \\ -4 & \text{if } x \ge 1 \end{cases}}$

2. Using the geometric meaning of the absolute value, draw the sets of points satisfying the conditions below. Write down the solutions of equations or inequalities.

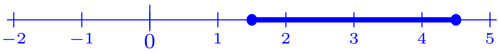
   (a) $|x+4| = 2$ $\boxed{x=-6 \text{ or } x=-2}$

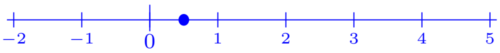
   (b) $|3x-2| > 1$ $x < \frac{1}{3}$ or $x > 1$

   (c) $|6-2x| \le 3$ $\boxed{\frac{3}{2} \le x \le \frac{9}{2}}$

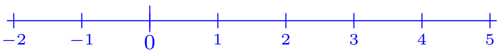
   (d) $|x+2| = |3-x|$ $\boxed{x = \frac{1}{2}}$

   (e) $|x+3| > |x-1|$ $\boxed{x > -1}$

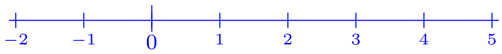
   (f) $|x|+|x-\sqrt{6}| = 1$ $\boxed{\text{no solutions}}$

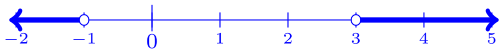
   (g) $|x+1|+|x-2| = 3$ $-1 \le x \le 2$

(h) $|x-5|+|x|<5$ $\boxed{\text{no solutions}}$



(i) $|x+1|+|x-3|>4$ $\boxed{x<-1 \text{ or } x>3}$



3. Write down the sets below using the absolute value notation $|\cdot|$.

(a) $\{4,18\}$ $\boxed{\big|x-11\big|=7}$

(b) $\{1+\sqrt{3},3+\sqrt{3}\}$ $\boxed{\big|x-(2+\sqrt{3})\big|=1}$

(c) $-3<x<3$ $\boxed{\big|x\big|<3}$

(d) $0\leq x\leq 2\sqrt{5}$ $\boxed{\big|x-\sqrt{5}\big|\leq\sqrt{5}}$

(e) $x\in(-\infty,4)\cup(10,+\infty)$ $\boxed{\big|x-7\big|>3}$

(f) $x\in(-\infty,-\sqrt{2}]\cup[2+\sqrt{2},+\infty)$ $\boxed{\big|x-1\big|\geq 1+\sqrt{2}}$

4. Solve equations and inequalities:

(a) $|x|+\sqrt{2}=|x+\sqrt{2}|$ $\boxed{x\geq 0}$

(b) $|x+1|+|x-2|=5$ $\boxed{x=-2 \text{ or } x=3}$

(c) $|3x+1|=|3-x|$ $\boxed{x=-2 \text{ or } x=\frac{1}{2}}$

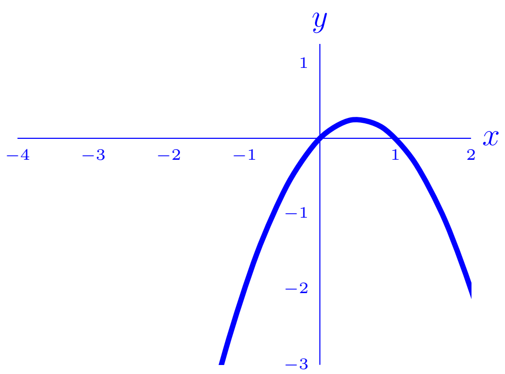
(d) $|x-2|<x$ $\boxed{x>1}$

(e) $|3-3x|\geq 6-3x$ $\boxed{x\geq\frac{3}{2}}$

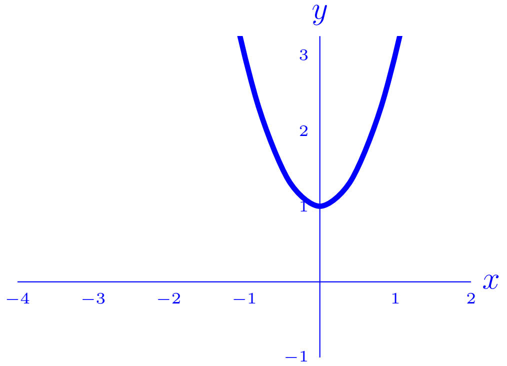
(f) $|1-2x|-|x+3|>x+4$ $\boxed{x<\frac{-3}{2}}$

5. Write the quadratic functions below in the product form (if it exists) and in the canonical form; draw the graphs:

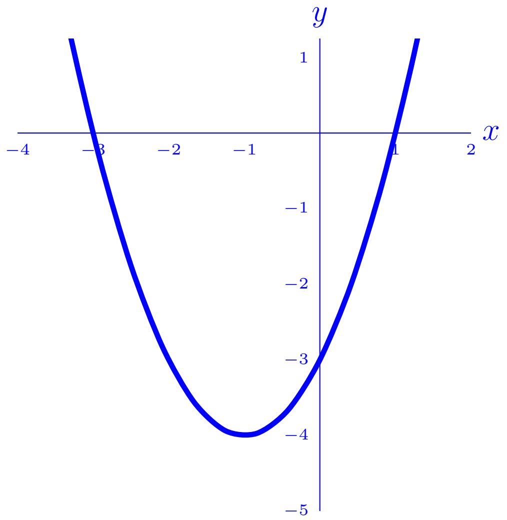
(a) $-x^2+x$ Product form: $(1-x)x$. Vertex form: $-(x-\frac{1}{2})^2+\frac{1}{4}$.



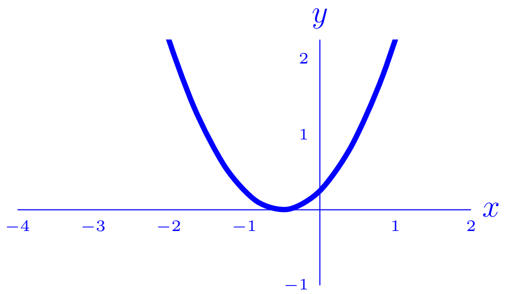
(b) $2x^2 + 1$ Product form does not exist with real numbers. $2x^2 + 1$ is vertex form.



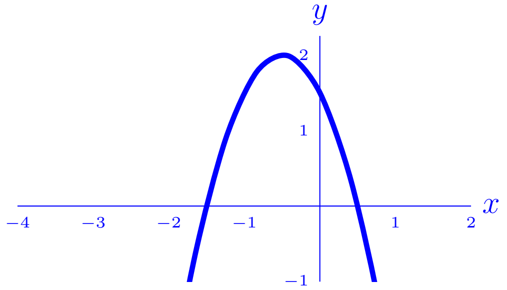
(c) $x^2 + 2x - 3$ Product form: $(x-1)(x+3)$. Vertex form: $(x+1)^2 - 4$.



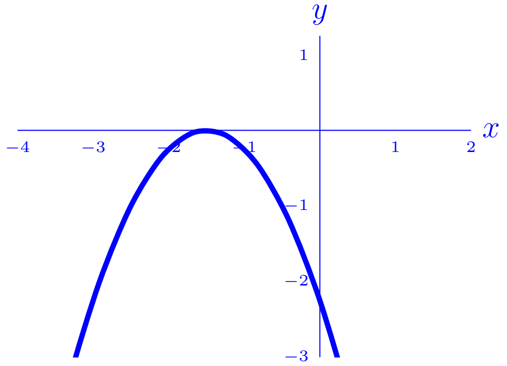
(d) $x^2 + x + \frac{1}{4}$ $\boxed{(x+\frac{1}{2})^2}$ is both product and vertex form.



(e) $-2x^2 - 2x + \frac{3}{2}$ Product form: $-2(x-\frac{1}{2})(x+\frac{3}{2})$. Vertex form: $-2(x+\frac{1}{2})^2 + 2$.



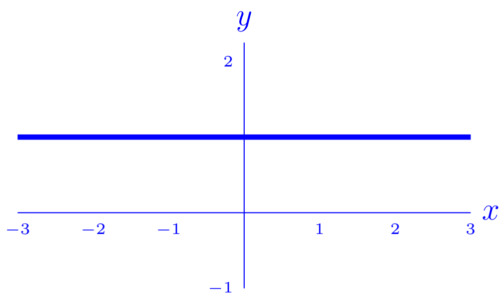
(f) $-x^2 - 3x - \frac{9}{4}$ $\boxed{-(x+\frac{3}{2})^2}$ is both product and vertex form.



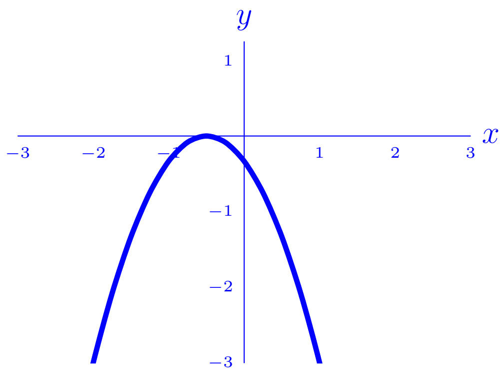
6. Determine the values of the parameter m in the function

$$f(x) = (m-3)x^2 + (m-3)x + m - 2$$

(a) so that the graph $y = f(x)$ is a straight line. Draw this graph. $\boxed{m = 3}$
$f(x) = 1$, so the graph is



(b) so that $f(x)$ has exactly one root. Draw the graph in this case also. $\boxed{m = \frac{5}{3}}$
$f(x) = -\frac{4}{3}x^2 - \frac{4}{3}x - \frac{1}{3} = \frac{-1}{3}(2x+1)^2.$



(c) so that the largest value of $f(x)$ is positive. $\boxed{\frac{5}{3} < m < 3}$

7. What are the values of m such that the function

$$f(x) = mx^2 + 4x + m - 3$$

(a) has a root, f has at least one root when $-1 \le m \le 4$. It has exactly one root when $m = -1$ or $m = 0$ (here f is linear) or $m = 4$.

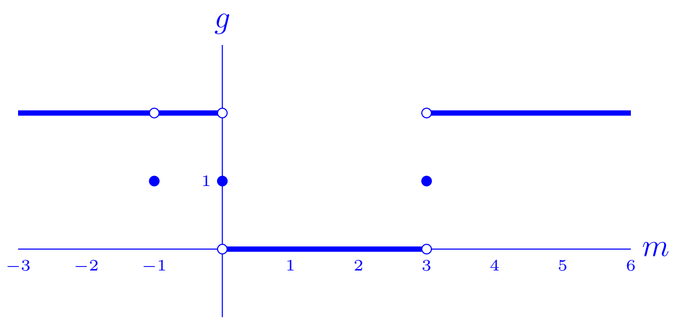
(b) has two roots of different signs, $\boxed{0 < m < 3}$

(c) has two positive roots, $\boxed{-1 < m < 0}$

(d) has a smallest value, and this value is positive. $\boxed{m > 4}$

8. Let $g(m)$ be the number of intersection points of a line $y = mx - 3$ and the graph of $y = (m+1)x^2 + (2-m)x - 2$, depending on m. Draw the graph of $g(m)$.

$$g(m) = \begin{cases} 0 & \text{if } m^2 - 3m < 0 \\ 1 & \text{if } m^2 - 3m = 0 \text{ or } m = -1 \\ 2 & \text{if } m^2 - 3m > 0 \text{ and } m \neq -1 \end{cases} = \begin{cases} 0 & \text{if } 0 < m < 3 \\ 1 & \text{if } m = -1 \text{ or } m = 0 \text{ or } m = 3 \\ 2 & \text{if } m < -1 \text{ or } -1 < m < 0 \text{ or } m > 3 \end{cases}$$



9. Find the coefficients and determine the degree of polynomials:

(a) $(x^4 - 3x^3 + x - 1)(x^2 - x + 4)$,
$\boxed{x^6 - 4x^5 + 7x^4 - 11x^3 - 2x^2 + 5x - 4}$ (degree $\boxed{6}$)

(b) $y = (x^3 + 5x^2 - x + 3)(x - 2)^2$,
$\boxed{x^5 + x^4 - 17x^3 + 27x^2 - 16x + 12}$ (degree $\boxed{5}$)

(c) $W(x) = (x + 2)^3 - (x - 1)^2$,
$\boxed{x^3 + 5x^2 + 14x + 7}$ (degree $\boxed{3}$)

(d) $y = (x + 1)^2 - (2x + 3)^3 - 2x$.
$\boxed{-8x^3 - 35x^2 - 54x - 26}$ (degree $\boxed{3}$)

10. Calculate the quotient and the remainder in the division of P by Q:

(a) $P(x) = 2x^4 - 3x^3 + 4x^2 - 5x + 6, \quad Q(x) = x^2 - 3x + 1$.
Quotient: $\boxed{2x^2 + 3x + 11}$. Remainder: $\boxed{25x - 5}$.
In other words, $\dfrac{2x^4 - 3x^3 + 4x^2 - 5x + 6}{x^2 - 3x + 1} = (2x^2 + 3x + 11) + \dfrac{25x - 5}{x^2 - 3x + 1}$.

(b) $P(x) = x^{16} - 16, \quad Q(x) = x^4 + 2$.
Quotient: $\boxed{x^{12} - 2x^8 + 4x^4 - 8}$. Remainder: $\boxed{0}$.

(c) $P(x) = x^5 - x^3 + 1, \quad Q(x) = (x - 1)^3$.
Quotient: $\boxed{x^2 + 3x + 5}$. Remainder: $\boxed{7x^2 - 12x + 6}$.

11. Find the value of a such that the remainder in the division of

$$W(x) = 2x^3 + (a^2 + 1)x^2 - (a + 2)x - 6$$

by $Q(x) = x + 3$ is as small as possible.

In general, the remainder of $W(x)$ after division by $x+3$ is equal to the value of $W(-3)$. In this case, $W(-3) = 9a^2 + 3a - 45$. Minimum value $-181/4$ occurs when $\boxed{a = -\frac{1}{6}}$.

12. Find all integer roots of the polynomials:
    (a) $x^3 + x^2 - 4x - 4$. The Rational Root Theorem says that roots $\pm p/q$ must satisfy $p|4$ and $q|1$. Only possibilities are $\pm1, \pm2, \pm4$. Actual integer roots are $\boxed{-2, -1, 2}$.
    (b) $3x^3 - 7x^2 + 4x - 4$. RRT says $p|4$ and $q|3$. For integers we only care about $q = 1$. Possibilities: $\pm1, \pm2, \pm4$. Actual roots: just $\boxed{2}$.
    (c) $x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6$. $\boxed{-2, 1, 3}$
    (d) $x^4 + 3x^3 - x^2 + 17x + 99$. $\boxed{\text{none}}$

13. Find all rational roots of the polynomials:
    (a) $4x^3 + x - 1$ $\boxed{\frac{1}{2}}$
    (b) $3x^4 - 8x^3 + 6x^2 - 1$ $\boxed{-\frac{1}{3}, 1}$
    (c) $x^3 - \frac{7}{6}x^2 - \frac{3}{2}x - \frac{1}{3}$ $\boxed{-\frac{1}{2}, -\frac{1}{3}, 2}$
    (d) $x^5 + \frac{4}{3}x^3 - x^2 + \frac{1}{3}x - \frac{1}{3}$ $\boxed{\text{None}}$

14. Write the polynomials below as products of irreducible components:
    ☆(a) x^6+8 $\boxed{(x^2 + 2)(x^2 + \sqrt{6}x + 2)(x^2 - \sqrt{6}x + 2)}$ The only way I (Adam) know to get this answer uses complex numbers, which are not part of this course.
    (b) $x^4 + x^2 + 1$ $\boxed{(x^2 - x + 1)(x^2 + x + 1)}$
    ☆(c) $x^4 - x^2 + 1$ $\boxed{(x^2 - \sqrt{3}x + 1)(x^2 - \sqrt{3}x + 1)}$ The only way I (Adam) know to get this answer uses complex numbers, which are not part of this course.
    (d) $4x^5 - 4x^4 - 13x^3 + 13x^2 + 9x - 9$ $\boxed{(2x - 3)(2x + 3)(x - 1)^2(x + 1)}$

15. Solve the equations:
    (a) $x^3 - 3x - 2 = 0$, $\boxed{x \in \{-1, 2\}}$. You could also write "$x = -1$ or $x = 2$".
    (b) $3x^4 - 10x^3 + 10x - 3 = 0$, $\boxed{x \in \{-1, \frac{1}{3}, 1, 3\}}$
    (c) $x^6 - 2\sqrt{2}x^3 + 2 = 0$, $\boxed{x \in \{2^{1/6}\}}$, which is the same as $x = 2^{1/6}$.
    (d) $x^4 - 2x^2 + 3x - 2 = 0$. $\boxed{x \in \{-2, 1\}}$

16. Solve the inequalities:
    (a) $x^3 - x^2 + 4x < 4$, $\boxed{x \in (-\infty, 1)}$
    (b) $x^3 - 6x^2 + 5x + 12 > 0$, $\boxed{x \in (-1, 3) \cup (4, \infty)}$
    (c) $(1 - x^2)(4x^2 + 8x - 21) \geq 0$, $\boxed{x \in [-\frac{7}{2}, -1] \cup [1, \frac{3}{2}]}$
    (d) $x^4 + 3x^3 + x^2 \leq 0$. $\boxed{x \in \{0\} \cup [\frac{-3-\sqrt{5}}{2}, \frac{-3+\sqrt{5}}{2}]}$

17. Solve the equations:

(a) $\dfrac{12}{1-9x^2} = \dfrac{1-3x}{1+3x} + \dfrac{1+3x}{3x-1}$ $\boxed{x=-1}$

(b) $\dfrac{30}{x^2-1} - \dfrac{13}{1+x+x^2} = \dfrac{7+18x}{x^3-1}$ $\boxed{x=-4, x=9}$

(c) $\dfrac{5}{x^2-4} + \dfrac{18}{x^2-3x+2} = \dfrac{8}{x^2-1}$ $\boxed{\text{no real solutions}}$

(d) $\dfrac{x}{x+a} + \dfrac{x}{x-a} = \dfrac{8}{3}$ $\boxed{x \in \{2a, -2a\}}$

18. Solve the inequalities:

(a) $\dfrac{(x-1)^2}{(x+1)^3} \leq 0$ $\boxed{x \in (-\infty, -1) \cup \{1\}}$

(b) $\dfrac{x^2+2}{x+1} < 2$ $\boxed{x \in (-\infty, -1) \cup (0, 2)}$

(c) $2 + \dfrac{3}{x+1} > \dfrac{2}{x}$ $\boxed{x \in (-\infty, -2) \cup (-1, 0) \cup (\frac{1}{2}, \infty)}$

(d) $\dfrac{1}{(x+1)^3} > \dfrac{1}{x+1}$ $\boxed{x \in (-\infty, -2) \cup (-1, 0)}$

(e) $\dfrac{x^2-5}{x} < x+1$ $\boxed{x \in (-5, 0)}$

(f) $\left|\dfrac{2x-3}{x-1}\right| \geq 2$ $\boxed{x \in (-\infty, 1) \cup (1, \frac{5}{4}]}$

(g) $\left|\dfrac{x^2-5x+3}{x^2-1}\right| < 1$ $\boxed{x \in (\frac{1}{2}, \frac{4}{5}) \cup (2, \infty)}$

(h) $\dfrac{x}{|x-2|} < 3$ $\boxed{x \in (-\infty, \frac{3}{2}) \cup (3, \infty)}$

(i) $\dfrac{\sqrt{x^2+6x+9}}{x} \geq -2$ $\boxed{x \in (-\infty, -1] \cup (0, \infty)}$

19. For which values of the parameters a and b does the equation

$$a + \frac{b}{x} = \frac{x-2}{x}$$

have a solution (for x)? $x = \dfrac{b+2}{1-a}$ is valid when $\boxed{a \neq 1}$.

Also answer this question for the equation

$$1 + \frac{b}{x} = \frac{x}{x-a}.$$

$x = \dfrac{ab}{b-a}$ is valid when $\boxed{a \neq b}$.

20. Prove that no integer can satisfy the inequality

$$\frac{1}{x} + \frac{1}{x+1} < \frac{2}{x+2}.$$

In order to avoid dividing by zero, x cannot be -2, -1, or 0. Thus solutions to the inequality might be in the intervals $(-\infty, -2)$, $(-2, -1)$, $(-1, 0)$, or $(0, \infty)$.

For $x \in (-\infty, -2)$, we have $x < 0$ and $x + 1 < 0$ and $x + 2 < 0$, which means $x(x+1)(x+2) < 0$. Therefore, multiplying the entire inequality by $x(x+1)(x+2)$ should reverse the direction of the inequality symbol:

$$\begin{aligned} \tfrac{1}{x} + \tfrac{1}{x+1} &< \tfrac{2}{x+2} \\ \tfrac{x(x+1)(x+2)}{x} + \tfrac{x(x+1)(x+2)}{x+1} &> \tfrac{2x(x+1)(x+2)}{x+2} \qquad \text{with } x < -2 \\ (x+1)(x+2) + x(x+2) &> 2x(x+1) \qquad \text{with } x < -2 \\ 2x^2 + 5x + 2 &> 2x^2 + 2x \qquad \text{with } x < -2 \\ 3x + 2 &> 0 \qquad \text{with } x < -2 \\ x &> -\tfrac{2}{3} \qquad \text{with } x < -2. \end{aligned}$$

There are no numbers that satisfy both $x > -\frac{2}{3}$ and $x < -2$, so there are solutions with $x \in (-\infty, -2)$.

The intervals $(-2, -1)$ and $(-1, 0)$ both do not contain any integers.

For $x \in (0, \infty)$, we have $x(x + 1)(x + 2) > 0$, so

$$(x + 1)(x + 2) + x(x + 2) < 2x(x + 1) \qquad \text{with } x > 0.$$

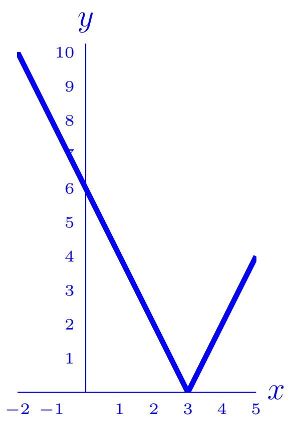
This simplifies to $x < -\frac{2}{3}$, but again there are no $x > 0$ that also satisfy $x < -\frac{2}{3}$.

Since there are no integer solutions in any of the intervals $(-\infty, -2)$, $(-2, -1)$, $(-1, 0)$, or $(0, \infty)$, there are no integer solutions to this inequality at all.

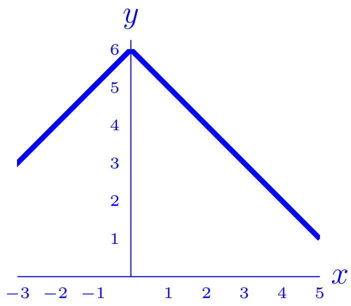
P.S. The task does not ask for it, but $(-2, -1) \cup (-\frac{2}{3}, 0)$ is the set of *real* solutions. There are no integers in this set.

21. Draw the graphs of functions:

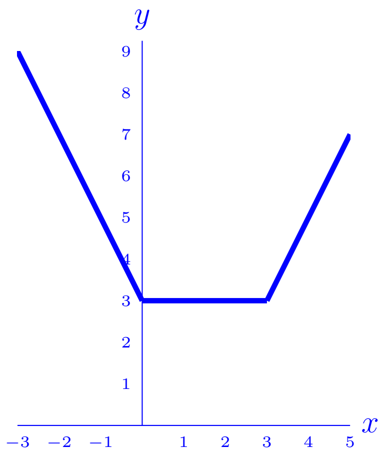
(a) $f(x) = |6 - 2x|$



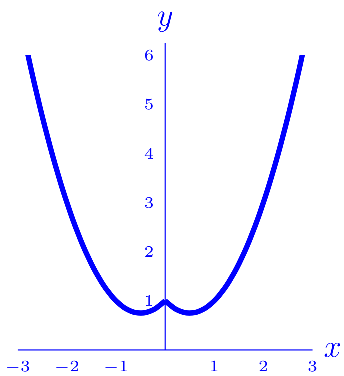
(b) $f(x) = 6 - |x|$



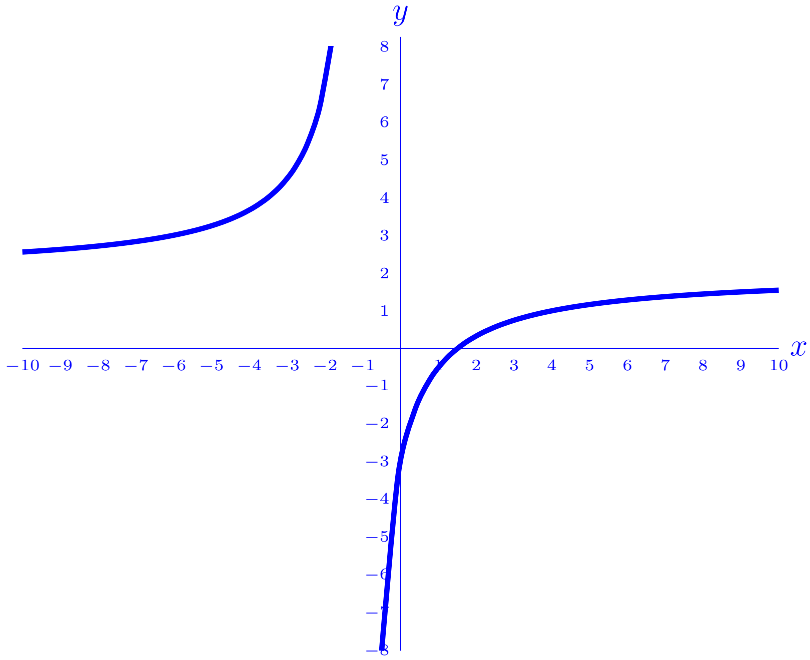
(c) $f(x) = \sqrt{x^2 - 6x + 9} + |x|$



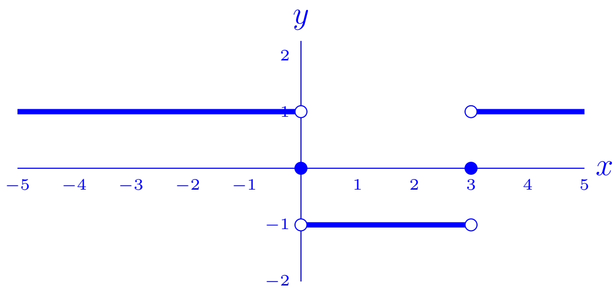
(d) $f(x) = x^2 - |x| + 1$



(e) $f(x) = \dfrac{2x-3}{x+1}$



(f) $f(x) = \operatorname{sgn}(x^2 - 3x)$.



Note: the function $\operatorname{sgn}(x)$ (the *sign* of x) takes the value $+1$ for $x > 0$, 0 for $x = 0$, and -1 for $x < 0$.