Math for Management, Winter 2023

List 4

Sequences, limits, asymptotes

67. (a) If $a_n = (n+2)^3$, give the value of a_3. $5^3 = \boxed{125}$

(b) For the sequence $b_n = n^{-n}$, what are the values b_1, b_2, and b_3?

$\boxed{b_1 = 1}$, $\boxed{b_2 = \frac{1}{4} = 0.25}$, $\boxed{b_3 = \frac{1}{27} \approx 0.037037}$

(c) If $c_n = (1+\frac{1}{n})^n$, what are the values c_1, c_2, and c_3? Give exact formulas (by hand) and decimal answers (using a calculator). $\boxed{c_1 = 2}$, $\boxed{c_2 = \frac{9}{4} = 2.25}$, $\boxed{c_3 = \frac{64}{27} \approx 2.3704}$

68. Consider the sequence

$$s_1 = 2$$
$$s_2 = 22$$
$$s_3 = 222$$
$$s_4 = 2222$$
$$s_n = \underbrace{22...2}_{n \text{ digits}}$$

(a) Calculate $(10s_1 + 2) - s_1$, then $(10s_2 + 2) - s_2$, then $(10s_3 + 2) - s_3$. $\boxed{20, \ \ 200, \ \ 2000}$

(b) Find a formula for $(10s_n + 2) - s_n$ in terms of n only. $\boxed{2 \cdot 10^n}$

(c) Find a formula for s_n. $\boxed{\frac{2}{9}(10^n - 1)}$

A sequence a_n is **monotonically increasing** if $a_{n+1} > a_n$ for all n. A sequence a_n is **monotonically decreasing** if $a_{n+1} < a_n$ for all n. A sequence is **monotonic** if it is either monotonically increasing or m. decreasing.

69. Label each of the following sequences as "monotonically increasing" or "monotonically decreasing" or "neither". Assume $n \geq 1$.

(a) n^2 $\boxed{\text{increasing}}$

(b) $\frac{2}{n^2}$ $\boxed{\text{decreasing}}$

(c) $(-5)^n$ $\boxed{\text{neither}}$

(d) $(-5)^{2n}$ $\boxed{\text{increasing}}$

(e) $\frac{n^3}{n^4+20}$ $\boxed{\text{neither}}$ because $a_2 > a_1$ but $a_4 < a_3$.

A sequence $(a_1, a_2, ...)$ is **arithmetic** if $a_{n+1} - a_n$ is constant. A sequence $(a_1, a_2, ...)$ is **geometric** if a_{n+1}/a_n is constant.

70. Find the general formula for the arithmetic sequence that satisfies $a_3 = 3$ and $a_{12} = 21$. Also calculate $S_{20} = a_1 + a_2 + \cdots + a_{20}$. $\boxed{a_n = -3 + 2n}$ and $\boxed{S_{20} = 360}$

71. Find the general formula for the geometric sequence that satisfies $a_2 = 18$ and $a_4 = 2$. Also calculate S_5. $\boxed{a_n = 162 \cdot (\frac{1}{3})^n}$ and $\boxed{S_5 = \frac{242}{3}}$

72. Find the sum of all three-digit numbers that are divisible by 3. $\boxed{165150}$

We say that **limit** of a sequence a_n is the number L and write

$$\text{“} \lim_{n\to\infty} a_n = L \text{”}$$

if for any $\varepsilon > 0$ there exists an N such that

$$L - \varepsilon < a_n < L + \varepsilon \qquad \text{for all } n > N.$$

We write “ $\lim_{n\to\infty} a_n = \infty$ ” if for any $M > 0$ there exist an N such that

$$a_n > M \qquad \text{for all } n > N.$$

Similarly, “ $\lim_{n\to\infty} a_n = -\infty$ ” if for any $M > 0$, ... $a_n < -M$ for all $n > N$.

73. (a) For which positive integers n is $4 - \dfrac{1}{100} < \dfrac{8n}{2n+9} < 4 + \dfrac{1}{100}$? $\boxed{n \geq 1796}$

(b) For which positive integers n is $\dfrac{8n}{2n+9} = 4$? $\boxed{\text{None!}}$

(c) Is it true that $\lim_{n\to\infty} \dfrac{8n}{2n+9} = 4$? $\boxed{\text{Yes}}$

74. Calculate $\lim_{n\to\infty} \dfrac{3n^2 + n + \sqrt{n}}{5n^2} = \boxed{\frac{3}{5}}$

75. Find the following limits if they exist.

(a) $\dfrac{n}{n+1}$ $\boxed{\text{yes}}$

(b) $(-1)^n$ $\boxed{\text{no}}$

(c) $\dfrac{3n}{9n+7}$ $\boxed{\text{yes}}$

☆(d) $\sin(3n)$ $\boxed{\text{no}}$

(e) $\sin(\pi n)$ $\boxed{\text{yes}}$ because the sequence is $0, 0, 0, 0, \ldots$

(f) $\dfrac{(-1)^{n+1}}{n}$ $\boxed{\text{yes}}$ Specifically, the limit is 0.

(g) $\lim_{n\to\infty} \dfrac{n+13}{n^2} = \boxed{0}$

(h) $\lim_{n\to\infty} \dfrac{(n+5)(n-2)}{n^2 - 6n + 7} = \boxed{1}$

(i) $\lim_{n\to\infty} \dfrac{n^2}{n+13} = \boxed{\infty}$

(j) $\lim_{n\to\infty} \frac{8}{\sqrt{n}} = \boxed{0}$

(k) $\lim_{n\to\infty} -2^n = \boxed{-\infty}$

(l) $\lim_{n\to\infty} (-2)^n$ $\boxed{\text{doesn't exist}}$

(m) $\lim_{n\to\infty} 2^{-n} = \boxed{0}$

(n) $\lim_{n\to\infty} 2^{1/n} = \boxed{1}$

(o) $\lim_{n\to\infty} \left((9\sqrt{n} + \frac{1}{\sqrt{n}})^2 - 81n\right) = \boxed{18}$

☆76. Find $\lim_{n\to\infty} n\cdot(2^{1/n} - 1)$. The ☆ means that this task is harder than what is normally expected in this course. $\boxed{\ln(2)}$

77. (a) Simplify the formula $\frac{\left(\sqrt{n}-\sqrt{n-1}\right)\left(\sqrt{n}+\sqrt{n-1}\right)}{\sqrt{n}+\sqrt{n-1}} = \boxed{\frac{1}{\sqrt{n}+\sqrt{n-1}}}$

(b) Find $\lim_{n\to\infty} \sqrt{n} - \sqrt{n-1} = \lim_{n\to\infty} \frac{1}{\sqrt{n}+\sqrt{n-1}} = \boxed{0}$

78. Use the Squeeze Theorem with $\frac{-1}{n} \le \frac{\cos(n)}{n} \le \frac{1}{n}$ to find $\lim_{n\to\infty} \frac{\cos(n)}{n}$.

$\lim_{n\to\infty} \frac{-1}{n} = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$, so by Squeeze Theorem we have $\lim_{n\to\infty} \frac{\cos(n)}{n} = \boxed{0}$.

☆79. Use the fact that $\left(1 - \frac{1}{\sqrt{n}}\right)^n \le \frac{1}{n}$ to find $\lim_{n\to\infty} (1/n)^{1/n}$.

We need an inequality involving $(1/n)^{1/n}$, but the right side of $\left(1-\frac{1}{\sqrt{n}}\right)^n \le \frac{1}{n}$ is just $(1/n)$. Raising both sides of the equation to the power $1/n$ gives

$$1 - \frac{1}{\sqrt{n}} \le \left(\frac{1}{n}\right)^{1/n}.$$

The Squeeze Theorem requires two inequalities. The left-hand side now has limit

$$\lim_{n\to\infty} 1 - \frac{1}{\sqrt{n}} = 1 - 0 = 1,$$

so another inequality involving a limit of 1 would be good. In fact,

$$\left(\frac{1}{n}\right)^{1/n} \le 1$$

is enough, and it is true because $\frac{1}{n} \le 1^n$ is true for all $n \ge 1$ (this is just $\frac{1}{n} \le 1$).

We can now use the Squeeze Theorem:

$$\begin{aligned} 1-\frac{1}{\sqrt{n}} &\le \left(\frac{1}{n}\right)^{1/n} \le 1 \\ \lim_{n\to\infty} 1-\frac{1}{\sqrt{n}} &\le \lim_{n\to\infty}\left(\frac{1}{n}\right)^{1/n} \le \lim_{n\to\infty} 1 \\ 1 &\le \lim_{n\to\infty}\left(\frac{1}{n}\right)^{1/n} \le 1 \\ \lim_{n\to\infty}\left(\frac{1}{n}\right)^{1/n} &= \boxed{1} \end{aligned}$$

80. (a) The *definition* of the number "0.385" is

$$3\cdot 10^{-1} \quad + \quad 8\cdot 10^{-2} \quad + \quad 5\cdot 10^{-2}.$$

Write this number as a fraction (or an integer, if possible). $\boxed{\frac{385}{1000} \text{ or } \frac{77}{200}}$

(b) The *definition* of the number "0.2222..." is the ***limit*** of the sequence

$$\begin{aligned} S_1 &= 0.2 \\ S_2 &= 0.22 \\ S_3 &= 0.222 \\ S_4 &= 0.2222 \\ S_n &= 0.\underbrace{22...2}_{n \text{ digits}} \end{aligned}$$

Write this number as a fraction (or an integer, if possible).
Hint: See Task 68(c).

$S_n = \dfrac{a_n \text{ from Task 68(c)}}{10^n} = \dfrac{\frac{2}{9}(10^n-1)}{10^n} = \frac{2}{9}(1-10^{-n}).$

Therefore $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2}{9}(1-10^{-n}) = \boxed{\frac{2}{9}}$

(c) The *definition* of the number "0.9999..." is the ***limit*** of the sequence

$$S_n = 0.\underbrace{99...9}_{n \text{ digits}}.$$

Write this number as a fraction (or an integer, if possible).

$S_n = 1-10^{-n}$, so $\lim_{n\to\infty} S_n = \boxed{1}$

81. Convert $1.8888\ldots = \boxed{\frac{17}{9}}$ and $0.313131\ldots = \boxed{\frac{31}{99}}$ into fractions.

82. Use the facts

$$0 < \ln(n) \qquad \text{for all } n\in\mathbb{N} \text{ with } n \ge 2$$

and

$$\ln(n) < \sqrt{n} \qquad \text{for all } n\in\mathbb{N}$$

to determine the value of $\lim_{n\to\infty} \frac{\ln(n)}{n}$. Diving the given inequalities by n (which is positive) gives $0 < \frac{\ln(n)}{n}$ and $\frac{\ln(n)}{n} < \frac{\sqrt{n}}{n}$. Using basic algebra,

$$\frac{\sqrt{n}}{n} = \frac{n^{1/2}}{n} = n^{-1/2} = \left(\frac{1}{n}\right)^{1/2},$$

so $\lim_{n\to\infty} \frac{\sqrt{n}}{n} = \left(\lim_{n\to\infty} \frac{1}{n}\right)^{1/2} = 0$, and the Squeeze Theorem gives $\lim_{n\to\infty} \frac{\ln(n)}{n} = \boxed{0}$.

83. Use the Squeeze Theorem to find $\lim_{n\to\infty} (5^n + 3^n)^{1/n}$ and $\lim_{n\to\infty} \frac{n^3}{3^n}$. First part: $\boxed{5}$ from

$$(5^n)^{1/n} \leq (5^n + 3^n)^{1/n} \leq (5^n + 5^n)^{1/n}.$$

Second part: $\boxed{0}$

84. Find the limits of these sequences and functions:

(a) $\lim_{n\to\infty} \frac{2^n + 4^{n+1/2}}{4^n} = \boxed{2}$

(b) $\lim_{x\to\infty} \frac{2^x + 4^{x+1/2}}{4^x} = \boxed{2}$

(c) $\lim_{n\to\infty} \frac{n^3 + n^{-3}}{n^2 + n^{-9}} = \boxed{\infty}$

(d) $\lim_{x\to\infty} \frac{x^3 + x^{-3}}{x^2 + x^{-9}} = \boxed{\infty}$

(e) $\lim_{n\to\infty} \sin(\pi n) = \boxed{0}$ because $\sin(\pi n) = 0$ for all $n \in \mathbb{N}$

(f) $\lim_{x\to\infty} \sin(\pi x)$ $\boxed{\text{doesn't exist}}$

85. Calculate $\lim_{x\to\infty} 6^x = \boxed{\infty}$ and $\lim_{x\to-\infty} 6^x = \boxed{0}$.

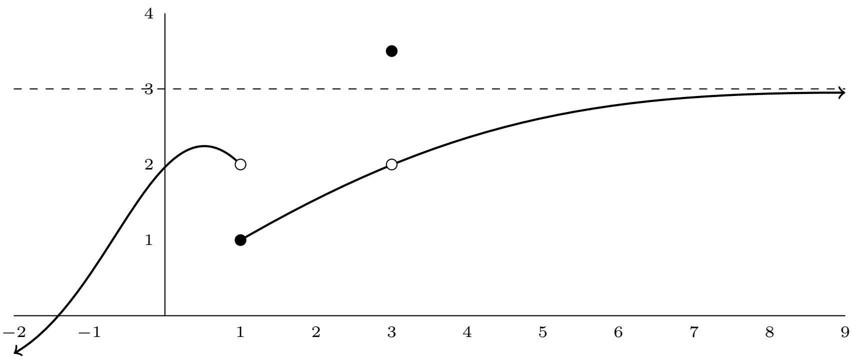
86. For the function whose graph is shown below, give the following limits (if they exist) to the nearest 0.5.

(a) $\lim_{x\to 1} f(x)$ $\boxed{\text{does not exist}}$

(b) $\lim_{x\to 2} f(x)$ $\boxed{1.5}$ (or something similar)

(c) $\lim_{x\to 3} f(x)$ $\boxed{2}$ (*not* 3.5, although $f(3) = 3.5$)

(d) $\lim_{x\to\infty} f(x)$ $\boxed{3}$



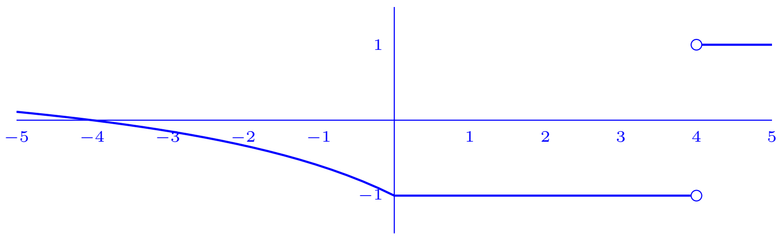
87. Does $\lim_{x\to 0} \frac{|x|-4}{|x-4|}$ exist? Yes Does $\lim_{x\to 4} \frac{|x|-4}{|x-4|}$ exist? No Draw a graph of the function for x-values between -5 and 5.

At $x = 0$, $f = \frac{0-4}{4} = -1$. At $x = 4$ the function is not defined. There are three regions to consider:

- $x > 4$ (in which $|x| = x$ and $|x-4| = x-4$),
- $0 < x < 4$ (in which $|x| = x$ but $|x-4| = 4-x$),
- $x < 0$ (in which $|x| = -x$ and $|x-4| = 4-x$)

In fact, we can write this as a piecewise function:

$$\frac{|x|-4}{|x-4|} = \begin{cases} \frac{-x-4}{4-x} & \text{if } x < 0 \\ \frac{x-4}{4-x} & \text{if } 0 \leq x < 4 \\ \frac{x-4}{x-4} & \text{if } x > 4 \end{cases} = \begin{cases} \frac{x+4}{x-4} & \text{if } x < 0 \\ -1 & \text{if } 0 \leq x < 4 \\ 1 & \text{if } x > 4 \end{cases}$$



88. Using the function $g(x) = \begin{cases} x^2 & \text{if } x \leq -2 \\ x & \text{if } -2 < x < 2, \\ 4 & \text{if } x = 2 \\ 3^{-x} & \text{if } x > 2 \end{cases}$ calculate the following:

(a) $\lim_{x\to -\infty} g(x) = \lim_{x\to -\infty} x^2 = \boxed{+\infty}$, or you can say it does not exist

(b) $\lim_{x\to (-2)^-} g(x) = \lim_{x\to -2^-} x^2 = \boxed{4}$

(c) $\lim_{x\to (-2)^+} g(x) = \lim_{x\to -2^+} x = \boxed{-2}$

(d) $\lim_{x\to -2} g(x)$ does not exist because $4 \neq -2$

(e) $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} x = \boxed{2}$

(f) $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} 3^{-x} = \boxed{0}$

89. Calculate $\lim_{t\to 8} \frac{t+4+t^{1/3}}{t^2-8t+7}$ and $\lim_{t\to -3} \frac{\sqrt{2t+22}-4}{t+3}$.

First part: just plug in $t=8$! $\frac{8+4+2}{64-64+7} = \frac{14}{7} = \boxed{2}$.

Second part:

$$\lim_{t\to -3} \frac{\sqrt{2t+22}-4}{t+3} = \lim_{t\to -3} \frac{\left(\sqrt{2t+22}-4\right)}{(t+3)} \frac{\left(\sqrt{2t+22}+4\right)}{\left(\sqrt{2t+22}+4\right)} = \lim_{t\to -3} \frac{2t+22-16}{(t+3)\left(\sqrt{2t+22}+4\right)}$$

$$= \lim_{t\to -3} \frac{2(t+3)}{(t+3)\left(\sqrt{2t+22}+4\right)} = \lim_{t\to -3} \frac{2}{\sqrt{2t+22}+4} = \frac{2}{8} = \boxed{\frac{1}{4}}$$

90. (a) Expand $(\sqrt{h+1}-1)(\sqrt{h+1}+1)$ and then simplify as much as possible.

$(\sqrt{h+1}-1)(\sqrt{h+1}+1) = (h+1)^2 - \sqrt{h+1} - \sqrt{h+1} - 1 = \boxed{h}$

(b) Calculate $\lim_{h\to 0} \frac{\sqrt{h+1}-1}{h}$.

$\lim_{h\to 0} \frac{\sqrt{h+1}-1}{h} \cdot \frac{\sqrt{h+1}+1}{\sqrt{h+1}+1} = \lim_{h\to 0} \frac{h}{h\sqrt{h+1}+h} = \lim_{h\to 0} \frac{1}{\sqrt{h+1}+1} = \boxed{\frac{1}{2}}$

91. Calculate the following limits:

(a) $\lim_{x\to\infty} \frac{3x^3-2x+1}{6x^3+x^2+x+19} = \boxed{\frac{1}{2}}$

(b) $\lim_{x\to\infty} \frac{3x^2-2x+1}{6x^3+x^2+x+19} = \boxed{0}$

(c) $\lim_{x\to 0} \left(\frac{8x-1}{x-x^2} + \frac{1}{x}\right) = \boxed{7}$

(d) $\lim_{x\to\infty} \left(\sqrt{9x^2+5x}-3x\right) = \boxed{\frac{5}{6}}$

(e) $\lim_{x\to\infty} (4^x+1)^{1/4} = \boxed{\infty}$

(f) $\lim_{x\to\infty} (4^x+x)^{1/x} = \boxed{4}$

(g) $\lim_{x\to 7} \frac{x^2-4x-21}{x^2-11x+28} = \boxed{\frac{10}{3}}$

(h) $\lim_{x\to 0} \frac{x^3-8x^2+3x+5}{x^9-6x^5+x^4-12x+1} = \boxed{5}$

92. (a) Find the vertical asymptote(s) of

$$g(x) = \frac{1}{x^2+x-6}.$$

$\boxed{x=-3, x=2}$

(b) Find the vertical asymptote(s) of

$$f(x) = \frac{x^2-x-2}{x^2+x-6}.$$

$\boxed{x=-3 \text{ only}}$

93. What horizontal asymptotes does the function

$$f(x) = \frac{x}{|x|+5}$$

have? Hint: Calculate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. $\boxed{y=1, y=-1}$

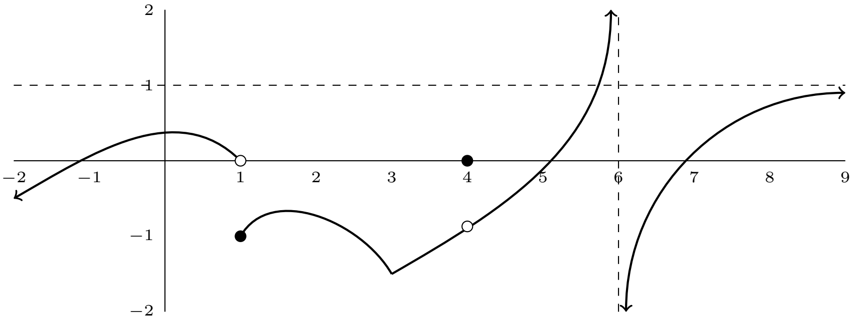
94. If $f(x)$ is a function for which

$$24x - 41 \quad \leq \quad f(x) \quad \leq \quad 4x^2 - 5$$

for all x, what is $\lim_{x\to 3} f(x)$?

$\lim_{x\to 3}(24x-41) = 31$ and $\lim_{x\to 3}(4x^2-5) = 31$, so the Squeeze Theorem guarantees that $\lim_{x\to 3} f(x) = \boxed{31}$.

95. List all points where the function graphed below is discontinuous.



$\boxed{x = 1, x = 4, x = 6}$ The function *is* continuous at $x = 3$.

96. Give an example of a function that is discontinuous at infinitely many points.

There are many examples. Here are two:

- $\boxed{\tan(x)}$ is discontinuous (in fact, undefined) at all $x = \dfrac{\pm\pi}{2} + 2\pi n$ for integer n.
- The "floor" function $\lfloor x \rfloor$ is discontinuous at every integer $x = n$.

☆97. Give an example of a function that is discontinuous at *every* point.

The "Dirichlet function" is a famous (well, famous within mathematics) example: $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$

98. Find all value(s) of the parameter p for which

$$f(x) = \begin{cases} 3x + p & \text{if } x \leq 8 \\ 2x - 5 & \text{if } x > 8 \end{cases}$$

is continous. $\boxed{p = -13}$

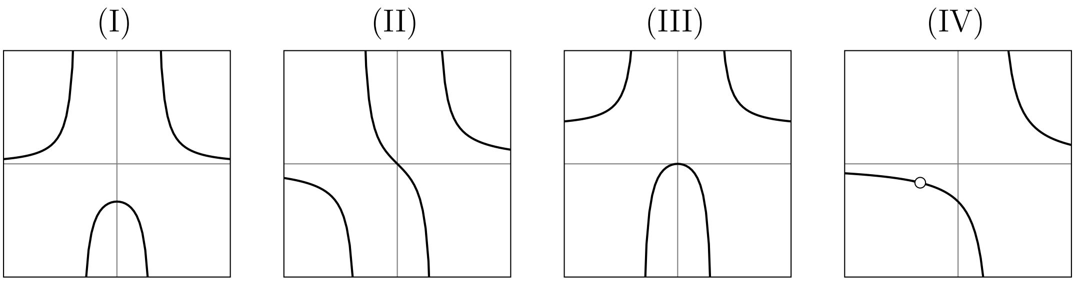
99. Find all value(s) of the parameters a, b for which

$$f(x) = \begin{cases} x & \text{if } |x| \leq 2 \\ x^2 + ax + b & \text{if } |x| > 2 \end{cases}$$

is continous. $\boxed{a = 1, b = -4}$

100. Match the functions with their graphs:

(a) $\dfrac{x}{x^2-1}$ $\boxed{\text{(II)}}$ (b) $\dfrac{1}{x^2-1}$ $\boxed{\text{(I)}}$ (c) $\dfrac{x+1}{x^2-1}$ $\boxed{\text{(IV)}}$ (d) $\dfrac{x^2}{x^2-1}$ $\boxed{\text{(III)}}$



101. Without graphing, determine which one of the three equations below has a solution with $0 \le x \le 3$.

(A) $x^2 = 4^x$, (B) $x^3 = 5^x$, (C) $x^5 = 6^x$.

$\boxed{\text{(C)}}$ because the function $f(x) = x^5 - 6^x$ has $f(0) = -1$ and $f(3) = 27$. Since $-1 < 0 < 27$, by the Intermediate Value Theorem, there must exist an x in $[0, 3]$ such that $f(x) = 0$.

102. Let $f(x) = \dfrac{13x - 77}{x - 5}$.

(a) $f(4) = 25$ and $f(11) = 11$. Does the Intermediate Value Theorem guarantee that $f(x) = 10$ for some $x \in [4, 11]$?
$\boxed{\text{No}}$ because f is discontinuous at $x = 5$.

(b) $f(6) = 1$ and $f(11) = 11$. Does the Intermediate Value Theorem guarantee that $f(x) = 10$ for some $x \in [6, 11]$?
$\boxed{\text{Yes}}$ because f is continuous on $[6, 11]$. (In fact $f(9) = 10$, though the task does not ask for this.)

(c) $f(6) = 1$ and $f(8) = 9$. Does the Intermediate Value Theorem guarantee that $f(x) = 10$ for some $x \in [6, 8]$?
$\boxed{\text{No}}$ because 10 is not in the y-interval $[f(6), f(8)] = [1, 9]$.

103. (a) Find $\displaystyle\lim_{x\to 0} \frac{(5+x)^3 - 125}{x} = \boxed{75}$

(b) Find $\displaystyle\lim_{h\to 0} \frac{(5+h)^3 - 125}{h} = \boxed{75}$

(c) Find $\displaystyle\lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$. Your answer will be a formula with x. $\boxed{3x^2}$