Math for Management, Winter 2023

List 4

Sequences, limits of sequences and functions

67. (a) If $a_n = (n+2)^3$, give the value of a_3.
    (b) For the sequence $b_n = n^{-n}$, what are the values b_1, b_2, and b_3?
    (c) If $c_n = (1+\frac{1}{n})^n$, what are the values c_1, c_2, and c_3? Give exact formulas (by hand) and decimal answers (using a calculator).

68. Consider the sequence

$$\begin{aligned} s_1 &= 2 \\ s_2 &= 22 \\ s_3 &= 222 \\ s_4 &= 2222 \\ s_n &= \underbrace{22...2}_{n \text{ digits}} \end{aligned}$$

    (a) Calculate $(10s_1 + 2) - s_1$, then $(10s_2 + 2) - s_2$, then $(10s_3 + 2) - s_3$.
    (b) Find a formula for $(10s_n + 2) - s_n$ in terms of n only.
    (c) Find a formula for s_n.

A sequence a_n is **monotonically increasing** if $a_{n+1} > a_n$ for all n.
A sequence a_n is **monotonically decreasing** if $a_{n+1} < a_n$ for all n.

A sequence is **monotonic** if it is either monotonically increasing or monotonically decreasing.

69. Label each of the following sequences as "monotonically increasing" or "monotonically decreasing" or "neither". Assume $n \geq 1$.

    (a) n^2 (b) $\dfrac{2}{n^2}$ (c) $(-5)^n$ (d) $(-5)^{2n}$ (e) $\dfrac{n^3}{n^4+20}$

A sequence $(a_1, a_2, ...)$ is **arithmetic** if $a_{n+1} - a_n$ is constant.
A sequence $(a_1, a_2, ...)$ is **geometric** if a_{n+1}/a_n is constant.

70. Find the general formula for the arithmetic sequence that satisfies $a_3 = 3$ and $a_{12} = 21$. Also calculate $S_{20} = a_1 + a_2 + \cdots + a_{20}$.

71. Find the general formula for the geometric sequence that satisfies $a_2 = 18$ and $a_4 = 2$. Also calculate S_5.

72. Find the sum of all three-digit numbers that are divisible by 3.

We say that **limit** of a sequence a_n is the number L and write

$$\text{“} \lim_{n\to\infty} a_n = L \text{”}$$

if for any $\varepsilon > 0$ there exists an N such that

$$L - \varepsilon < a_n < L + \varepsilon \qquad \text{for all } n > N.$$

We write " $\lim_{n\to\infty} a_n = \infty$ " if for any $M > 0$ there exist an N such that

$$a_n > M \qquad \text{for all } n > N.$$

Similarly, " $\lim_{n\to\infty} a_n = -\infty$ " if for any $M > 0$, ... $a_n < -M$ for all $n > N$.

73. (a) For which positive integers n is $4 - \frac{1}{100} < \frac{8n}{2n+9} < 4 + \frac{1}{100}$?

    (b) For which positive integers n is $\frac{8n}{2n+9} = 4$?

    (c) Is it true that $\lim_{n\to\infty} \frac{8n}{2n+9} = 4$?

74. Calculate $\lim_{n\to\infty} \frac{3n^2 + n + \sqrt{n}}{5n^2}$.

75. Find the following limits if they exist.

    (a) $\lim_{n\to\infty} \frac{n}{n+1}$

    (b) $\lim_{n\to\infty} (-1)^n$

    (c) $\lim_{n\to\infty} \frac{3n}{9n+7}$

    ☆(d) $\lim_{n\to\infty} \sin(3n)$

    (e) $\lim_{n\to\infty} \sin(\pi n)$

    (f) $\lim_{n\to\infty} \frac{(-1)^{n+1}}{n}$

    (g) $\lim_{n\to\infty} \frac{n+13}{n^2}$

    (h) $\lim_{n\to\infty} \frac{(n+5)(n-2)}{n^2 - 6n + 7}$

    (i) $\lim_{n\to\infty} \frac{n^2}{n+13}$

    (j) $\lim_{n\to\infty} \frac{8}{\sqrt{n}}$

    (k) $\lim_{n\to\infty} -2^n$

    (l) $\lim_{n\to\infty} (-2)^n$

    (m) $\lim_{n\to\infty} 2^{-n}$

    (n) $\lim_{n\to\infty} 2^{1/n}$

    (o) $\lim_{n\to\infty} \left((9\sqrt{n} + \tfrac{1}{\sqrt{n}})^2 - 81n\right)$.

☆76. Find $\lim_{n\to\infty} n \cdot (2^{1/n} - 1)$. The ☆ means that this task is harder than what is normally expected in this course.

77. (a) Simplify the formula $\frac{\left(\sqrt{n} - \sqrt{n-1}\right)\left(\sqrt{n} + \sqrt{n-1}\right)}{\sqrt{n} + \sqrt{n-1}}$.

    (b) Find $\lim_{n\to\infty} \sqrt{n} - \sqrt{n-1}$.

78. Use the Squeeze Theorem with $\frac{-1}{n} \leq \frac{\cos(n)}{n} \leq \frac{1}{n}$ to find $\lim_{n\to\infty} \frac{\cos(n)}{n}$.

☆79. Use the fact that $\left(1 - \frac{1}{\sqrt{n}}\right)^n \leq \frac{1}{n}$ to find $\lim_{n\to\infty} (1/n)^{1/n}$.

80. (a) The *definition* of the number “0.385” is

$$3 \cdot 10^{-1} \quad + \quad 8 \cdot 10^{-2} \quad + \quad 5 \cdot 10^{-2}.$$

Write this number as a fraction (or an integer, if possible).

(b) The *definition* of the number “0.2222...” is the ***limit*** of the sequence

$$\begin{aligned} S_1 &= 0.2 \\ S_2 &= 0.22 \\ S_3 &= 0.222 \\ S_4 &= 0.2222 \\ S_n &= 0.\underbrace{22...2}_{n \text{ digits}} \end{aligned}$$

Write this number as a fraction (or an integer, if possible).
Hint: See Task 68(c).

(c) The *definition* of the number “0.9999...” is the ***limit*** of the sequence

$$S_n = 0.\underbrace{99...9}_{n \text{ digits}}.$$

Write this number as a fraction (or an integer, if possible).

81. Convert $1.8888\ldots$ and $0.313131\ldots$ into fractions.

82. Use the facts

$$0 < \ln(n) \qquad \text{for all } n \in \mathbb{N} \text{ with } n \geq 2$$

and

$$\ln(n) < \sqrt{n} \qquad \text{for all } n \in \mathbb{N}$$

to determine the value of $\lim_{n\to\infty} \frac{\ln(n)}{n}$.

83. Use the Squeeze Theorem to find $\lim_{n\to\infty} \left(5^n + 3^n\right)^{1/n}$ and $\lim_{n\to\infty} \frac{n^3}{3^n}$.

84. Find the limits of these sequences and functions:

(a) $\lim_{n\to\infty} \frac{2^n + 4^{n+1/2}}{4^n}$ (c) $\lim_{n\to\infty} \frac{n^3 + n^{-3}}{n^2 + n^{-9}}$ (e) $\lim_{n\to\infty} \sin(\pi n)$

(b) $\lim_{x\to\infty} \frac{2^x + 4^{x+1/2}}{4^x}$ (d) $\lim_{x\to\infty} \frac{x^3 + x^{-3}}{x^2 + x^{-9}}$ (f) $\lim_{x\to\infty} \sin(\pi x)$

85. Calculate $\lim_{x\to\infty} 6^x$ and $\lim_{x\to-\infty} 6^x$.

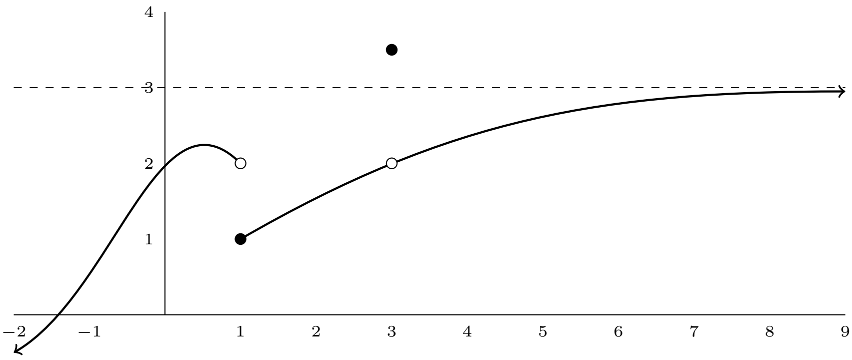
86. For the function whose graph is shown below, give the following limits (if they exist) to the nearest 0.5.

(a) $\lim_{x\to 1} f(x)$

(b) $\lim_{x\to 2} f(x)$

(c) $\lim_{x\to 3} f(x)$

(d) $\lim_{x\to\infty} f(x)$



87. Does $\lim_{x\to 0} \frac{|x|-4}{|x-4|}$ exist? Does $\lim_{x\to 4} \frac{|x|-4}{|x-4|}$ exist? Draw a graph of the function for x-values between -5 and 5.

88. Using the function $g(x) = \begin{cases} x^2 & \text{if } x \le -2 \\ x & \text{if } -2 < x < 2, \\ 4 & \text{if } x = 2 \\ 3^{-x} & \text{if } x > 2 \end{cases}$ calculate the following:

(a) $\lim_{x\to-\infty} g(x)$

(b) $\lim_{x\to(-2)^-} g(x)$

(c) $\lim_{x\to(-2)^+} g(x)$

(d) $\lim_{x\to-2} g(x)$

(e) $\lim_{x\to 2^-} g(x)$

(f) $\lim_{x\to\infty} g(x)$

89. Calculate $\lim_{t\to 8} \frac{t+4+t^{1/3}}{t^2-8t+7}$ and $\lim_{t\to-3} \frac{\sqrt{2t+22}-4}{t+3}$.

90. (a) Expand $(\sqrt{h+1}-1)(\sqrt{h+1}+1)$ and then simplify as much as possible.

(b) Calculate $\lim_{h\to 0} \frac{\sqrt{h+1}-1}{h}$.

91. Calculate the following limits:

(a) $\lim_{x\to\infty} \frac{3x^3-2x+1}{6x^3+x^2+x+19}$

(b) $\lim_{x\to\infty} \frac{3x^2-2x+1}{6x^3+x^2+x+19}$

(c) $\lim_{x\to 0} \left(\frac{8x-1}{x-x^2}+\frac{1}{x}\right)$

(d) $\lim_{x\to\infty} \left(\sqrt{9x^2+5x}-3x\right)$

(e) $\lim_{x\to\infty} (4^x+1)^{1/4}$

(f) $\lim_{x\to\infty} (4^x+x)^{1/x}$

(g) $\lim_{x\to 7} \frac{x^2-4x-21}{x^2-11x+28}$

(h) $\lim_{x\to 0} \frac{x^3-8x^2+3x+5}{x^9-6x^5+x^4-12x+1}$

92. (a) Find the vertical asymptote(s) of

$$g(x) = \frac{1}{x^2 + x - 6}.$$

(b) Find the vertical asymptote(s) of

$$f(x) = \frac{x^2 - x - 2}{x^2 + x - 6}.$$

93. What horizontal asymptotes does the function

$$f(x) = \frac{x}{|x| + 5}$$

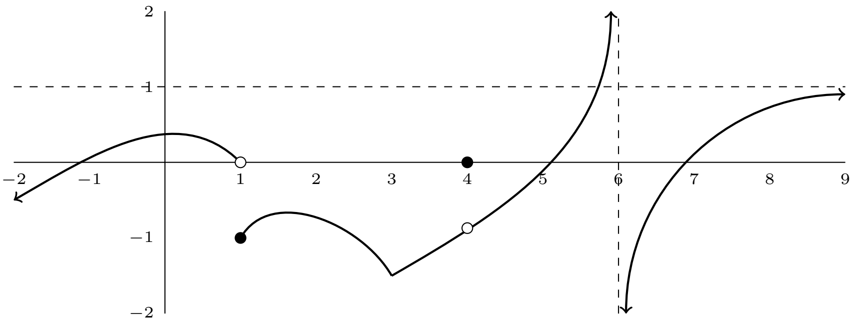
have? Hint: Calculate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

94. If $f(x)$ is a function for which

$$24x - 41 \quad \leq \quad f(x) \quad \leq \quad 4x^2 - 5$$

for all x, what is $\lim_{x\to 3} f(x)$?

95. List all points where the function graphed below is discontinuous.



96. Give an example of a function that is discontinuous at infinitely many points.

☆97. Give an example of a function that is discontinuous at *every* point.

98. Find all value(s) of the parameter p for which

$$f(x) = \begin{cases} 3x + p & \text{if } x \leq 8 \\ 2x - 5 & \text{if } x > 8 \end{cases}$$

is continous.

99. Find all value(s) of the parameters a, b for which

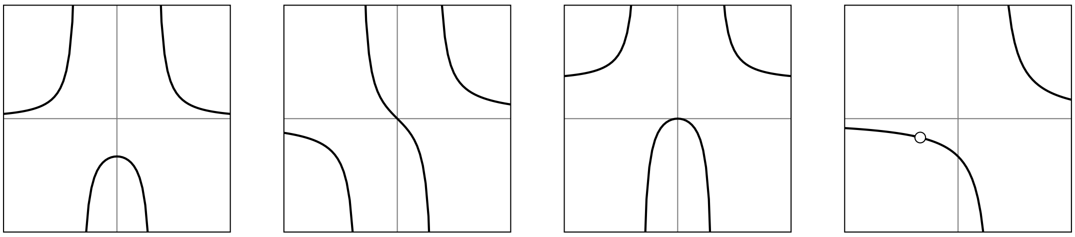
$$f(x) = \begin{cases} x & \text{if } |x| \leq 2 \\ x^2 + ax + b & \text{if } |x| > 2 \end{cases}$$

is continous.

100. Match the functions with their graphs:

(a) $\dfrac{x}{x^2-1}$ (b) $\dfrac{1}{x^2-1}$ (c) $\dfrac{x+1}{x^2-1}$ (d) $\dfrac{x^2}{x^2-1}$

(I) (II) (III) (IV)



101. Without graphing, determine which one of the three equations below has a solution with $0 \le x \le 3$.

(A) $x^2 = 4^x$, (B) $x^3 = 5^x$, (C) $x^5 = 6^x$.

102. Let $f(x) = \dfrac{13x-77}{x-5}$.

(a) $f(4) = 25$ and $f(11) = 11$. Does the Intermediate Value Theorem guarantee that $f(x) = 10$ for some $x \in [4, 11]$?

(b) $f(6) = 1$ and $f(11) = 11$. Does the Intermediate Value Theorem guarantee that $f(x) = 10$ for some $x \in [6, 11]$?

(c) $f(6) = 1$ and $f(8) = 9$. Does the Intermediate Value Theorem guarantee that $f(x) = 10$ for some $x \in [6, 8]$?

103. (a) Find $\displaystyle\lim_{x\to 0} \frac{(5+x)^3-125}{x}$.

(b) Find $\displaystyle\lim_{h\to 0} \frac{(5+h)^3-125}{h}$.

(c) Find $\displaystyle\lim_{h\to 0} \frac{(x+h)^3-x^3}{h}$. Your answer will be a formula with x.