Math for Management, Winter 2023

List 5

Derivative calculations

For a function $f(x)$ and a number a, the **derivative of f at a**, written $f'(a)$, is the slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$ and is calculated as

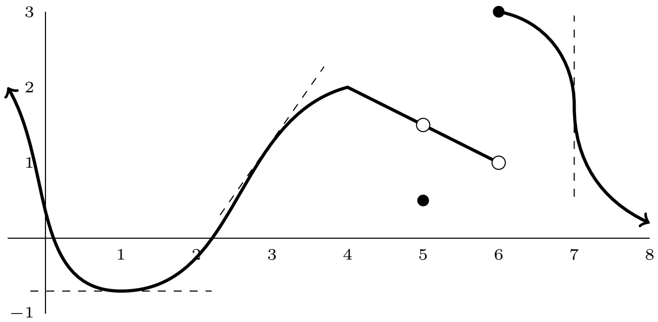
$$f'(a) \quad = \quad \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \quad = \quad \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

The function $f(x)$ is **differentiable at a** if $f'(a)$ exists and is finite.

104. Calculate $f'(5)$ for the function $f(x) = x^3$. Hint: See Task 90(b).

105. Calculate $f'(1)$ for the function $f(x) = \sqrt{x}$. Hint: See Task 50(b).

106. The graph of a function is shown below. Near $x = 1$, $x = 3$, and $x = 7$, part of the tangent lines to the graph at those points is shown as a dashed line segment.



(a) List all points where the function is not continuous.

(b) List all points where the function is not differentiable (that is, where the derivative does not exist).

107. List all points where $f(x) = \dfrac{|x| - 4}{|x - 4|}$ is not differentiable.

The Constant Multiple Rule: If c is a constant then

$$(c\,f)' = c\,f' \qquad \big(c\,f(x)\big)' = c\,f'(x) \qquad \tfrac{\mathrm{d}}{\mathrm{d}x}\big[c\,f\big] = c\,\tfrac{\mathrm{d}f}{\mathrm{d}x} \qquad D\big[c\,f\big] = c\,D[f]$$

(these are four ways of writing exact the same fact).

The Sum Rule: $\frac{\mathrm{d}}{\mathrm{d}x}\big[f + g\big] = \frac{\mathrm{d}}{\mathrm{d}x}[f] + \frac{\mathrm{d}}{\mathrm{d}x}[g]$.

The Power Rule: If p is a constant then $\frac{\mathrm{d}}{\mathrm{d}x}\big[x^p\big] = p\,x^{p-1}$.

108. All parts of this task have exactly the same answer!

(a) Find $f'(x)$ for the function $f(x) = 2x^7$.

(b) Give f' if $f = 2x^7$.

(c) Find y' for $y = 2x^7$.

(d) Compute $\frac{\mathrm{d}f}{\mathrm{d}x}$ for the function $f(x) = 2x^7$.

(e) Compute $\frac{\mathrm{d}y}{\mathrm{d}x}$ for $y = 2x^7$.

(f) Give the derivative of $2x^7$ with respect to x.

(g) Find the derivative of $2x^7$.

(h) Calculate $\frac{\mathrm{d}}{\mathrm{d}x}2x^7$. (i) Calculate $\left(2x^7\right)'$. (j) Calculate $D\left[2x^7\right]$.

(k) Differentiate $2x^7$ with respect to x.

(ℓ) Differentiate $2x^7$.

109. Differentiate $x^5 + \frac{2}{9}x^3 + \sqrt{3x} + \dfrac{x^{10}}{\sqrt{x}}$.

110. Differentiate $\left(x + \sqrt{x}\right)^2$.

111. For each of the functions below, can the Power Rule and/or Constant Multiple Rule (along with maybe some algebra) be used to find the derivative? If so, give the derivative.

(a) $2x^6$ (e) $x^{\sin x}$ (i) $\sin(5\cos(x))$ (m) $\ln(2+x)$

(b) $2\sqrt{x}$ (f) $(\sin x)^x$ (j) $e^{5\ln(x)}$ (n) $\ln(2x)$

(c) $\sqrt{5x}$ (g) e^x (k) $\dfrac{3}{x^6}$ (o) $\ln(2^x)$

(d) x^π (h) $\cos(5x)$ (ℓ) x^x (p) $\ln(x^2)$

112. Is it possible to find the derivative of the following functions using the Power Rule, Constant Multiple Rule, and Sum Rule?

(a) $x+\ln(5e^x)$ (b) $\dfrac{2x}{x+6}$ (c) $\dfrac{x+6}{2x}$ (d) $\dfrac{x+\frac{1}{x}}{\sqrt{x}}$

113. Give the derivative of each of the following functions.

(a) x^{7215}

(b) $5x^{100} + 9x$

(c) $2x^3 - 6x^2 + 10x + 1$

(d) $3\sqrt{x}$

(e) $\sqrt[3]{x}$

(f) $\sqrt{x}^3$

(g) 31

(h) $x + \frac{1}{x}$

(i) $\sqrt{x} + \frac{1}{\sqrt{x}}$

(j) $(3x+7)^2$

114. Is $x^3 - x^{1/3}$ continuous everywhere? Is it differentiable everywhere?

115. If $f(x) = 8x^4 - x^2$, for what values of x does $f(x) = 0$?
For what values of x does $f'(x) = 0$?

116. For the function $f(x) = x^3$ and $g(x) = 2x^2$, ...

(a) Calculate the derivative of f.

(b) Calculate the derivative of g.

(c) Calculate the derivative of

$$f(x) + g(x) = x^3 + 2x^2.$$

(d) Calculate the derivative of

$$f(x)\cdot g(x) = 2x^5.$$

(e) Does $(f+g)' = f' + g'$? In other words, is your answer to (c) the same as adding your answers to (a) and (b)?

(f) Does the derivative of a sum equal the sum of the derivatives?

(g) Does $(f\cdot g)' = f'\cdot g'$? In other words, is your answer to (d) the same as multiplying your answers to (a) and (b)?

(h) Does $\frac{\mathrm{d}}{\mathrm{d}x}\big[f\cdot g\big] = \frac{\mathrm{d}f}{\mathrm{d}x}\cdot\frac{\mathrm{d}g}{\mathrm{d}x}$?

(i) Does the derivative of a product equal the product of the derivatives?

117. Which limit expression below gives the derivative of x^3 at the point $x = 2$?

(A) $\displaystyle\lim_{x\to 2}\frac{x^3-8}{x}$ (C) $\displaystyle\lim_{h\to 0}\frac{(2+h)^3-8}{h}$

(B) $\displaystyle\lim_{h\to 0}\frac{h^3-8}{h}$ (D) $\displaystyle\lim_{h\to 0}\frac{(2+h)^3-h^3}{h}$

118. (a) Find $\big(x^{10}+100x+1000\big)'$.

(b) Find $D\big[9x+\sqrt{9x}\big]$.

(c) Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\Big[(2x+3)^2\Big]$.

(d) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for $y = \dfrac{x+12}{2x}$.

119. For each function below, state whether its derivative can be found using *only* algebra, the Power Rule, the Constant Multiple Rule, and the Sum Rule. If so, give its derivative.

(a) $4x^2-27x$

(b) $4x^2-27$

(c) $\sqrt{16x}$

(d) $(x+\sqrt{7})^2$

(e) 2^{x+7}

(f) $\dfrac{5}{x}$

(g) $\dfrac{3x}{6x+15}$

(h) $\dfrac{6x+15}{3x}$

Individual functions: $\big(x^p\big)' = px^{p-1}$, $\big(a^x\big)' = a^x\ln(a)$, $\big(\ln(x)\big)' = \frac{1}{x}$,

$\big(\sin(x)\big)' = \cos(x)$, $\big(\cos(x)\big)' = -\sin(x)$.

Sum Rule: $(f+g)' = f'+g'$ **Product Rule:** $(f\cdot g)' = fg'+f'g$

Chain Rule: $\big(f(g)\big)' = f'(g)\cdot g'$ **Quotient Rule:** $\left(\frac{f}{g}\right)' = \dfrac{gf'-fg'}{g^2}$

120. Give the derivative of $5\sin(x)+\frac{2}{3}\cos(x)-x^3+9$.

121. Using the Product Rule, give the derivative of $5^x\cdot\sin(x)$.

122. Use the Product Rule (twice) to find the derivative of $x^6\cdot\cos(x)\cdot 2^x$.

123. Give the derivative of every function in Task 119.

124. True or false?

(a) $(f+g)' = f' + g'$

(b) $(f \cdot g)' = f' \cdot g'$

(c) $(f \cdot g)' = f'g + fg'$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}(fg) = f\frac{\mathrm{d}g}{\mathrm{d}x} + g\frac{\mathrm{d}f}{\mathrm{d}x}$

(e) $(f \cdot g)' = g'f' + gf'$

(f) $(f/g)' = gf' - fg'$

125. Find the derivative of $\sin(5^{\cos(2x^3+8)})$.

126. (a) Use the Quotient Rule to differentiate $\dfrac{\sin(x)}{x^4}$.

(b) Use the Product Rule to differentiate $x^{-4}\sin(x)$.

(c) Use algebra to compare your answers from parts (a) and (b).

127. Find the following derivatives (note (p)-(ż) require the Chain Rule).

(a) $f'(x)$ for $f(x) = x^9 \sin(x)$

(ą) $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^9 \sin(x)\right)$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\left(10^x + \log_{10}(x)\right)$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(10^x \cdot \log_{10}(x)\right)$

(ć) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}\sin(x)\right)$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^9\, e^x \sin(x)\right)$

(e) $\frac{\mathrm{d}}{\mathrm{d}x}(4x^3 + x\sin x)$

(ę) $\frac{\mathrm{d}}{\mathrm{d}t}(4t^3 + t\sin t)$

(f) $\frac{\mathrm{d}}{\mathrm{d}t}\ \sin(t)\cos(t)$

(g) $\dfrac{\mathrm{d}}{\mathrm{d}x}\,\dfrac{\cos(x)}{5x^3 - 12}$

(h) $\dfrac{\mathrm{d}}{\mathrm{d}x}\,\dfrac{5x^3 - 12}{\cos(x)}$

(i) $\dfrac{\mathrm{d}}{\mathrm{d}t}\,\dfrac{t^7 + t^2}{e^t}$

(j) $\frac{\mathrm{d}}{\mathrm{d}x}(5x-7)^2$

(k) $\frac{\mathrm{d}}{\mathrm{d}t}\ e^t\cos(t)$

(l) $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(t\sin(t) + \dfrac{e^t}{t^2+1}\right)$

(ł) $\dfrac{\mathrm{d}}{\mathrm{d}x}\,\dfrac{\sin(t)}{t\,e^t}$

(m) $\frac{\mathrm{d}}{\mathrm{d}t}\ t^{5/2}\sin(t)$

(n) $\frac{\mathrm{d}}{\mathrm{d}x}\ 2^{15}$

(ń) $\frac{\mathrm{d}}{\mathrm{d}x}\ x^{15}$

(o) $\frac{\mathrm{d}}{\mathrm{d}u}\ u^{15}$

(ó) $\frac{\mathrm{d}}{\mathrm{d}x}\ u^{15}$ if u is a constant

(p) $\frac{\mathrm{d}}{\mathrm{d}x}\ u^{15}$ if u is a fn. of x

(q) $\frac{\mathrm{d}}{\mathrm{d}x}(\cos(x))^{15}$

(r) $\frac{\mathrm{d}}{\mathrm{d}x}\ln(\cos(x))$

(s) $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{\ln(\cos(x))}$

(ś) $\dfrac{\mathrm{d}}{\mathrm{d}x}\ e^{\sqrt{\ln(\cos(x))}}$

(t) $\dfrac{\mathrm{d}}{\mathrm{d}x}\ e^{\sqrt{\ln(\cos(x^6))}}$

(u) $\frac{\mathrm{d}}{\mathrm{d}t}\ 5\sin(2t+1)$

(v) $\frac{\mathrm{d}}{\mathrm{d}t}\ A\sin(\omega t + \phi)$ if A, ω, t are constants

(w) $\frac{\mathrm{d}}{\mathrm{d}x}(7x^2 + \sin(x))^2$

(x) $\frac{\mathrm{d}}{\mathrm{d}x}(\log_3(x))^2$

(y) $\frac{\mathrm{d}}{\mathrm{d}t}\tan(t^3 + 8t^2 + 2t + 18)$

(z) $\frac{\mathrm{d}}{\mathrm{d}x}\cos(x^3 e^x)$

(ź) $\frac{\mathrm{d}}{\mathrm{d}x}\ x^3\cos(9x)$

(ż) $\dfrac{\mathrm{d}}{\mathrm{d}x}\,\dfrac{x^3\cos(x)}{e^{\sin(x)}}$