Math for Management, Winter 2023

List 6

Derivative applications (monotonicity, convexity, min/max)

128. Calculate $f'(2)$ for the function $f(x) = x^4 + 4x$. $\boxed{36}$

129. Find the *slope* of the tangent line to $y = x^4 + 4x$ at the point $(2, 24)$. This is exactly the same as Task 128! $\boxed{36}$

130. Give an *equation* for the tangent line to $y = x^4 + 4x$ through the point $(2, 24)$. The line through $(2, 24)$ with slope 36 can be described by $\boxed{y = 24 + 36(x-2)}$ or by $\boxed{y = 36x - 48}$ or other formats.

131. Give an equation for the tangent line to $y = \dfrac{1}{\sqrt{x}}$ at $x = 4$. $y = x^{-1/2}$, so $y' = \frac{-1}{2}x^{-3/2}$, so the slope is $y'(4) = \frac{-1}{2}(4)^{-3/2}\frac{-1}{2}(2)^{-3} = \frac{-1}{2} \cdot \frac{1}{8} = \frac{-1}{16}$. The line through $(4, \frac{1}{2})$ with slope $\frac{-1}{16}$ is $\boxed{y = \frac{1}{2} - \frac{1}{16}(x-4)}$ or $\boxed{y = \frac{-1}{16}x + \frac{3}{4}}$.

132. Give an equation for the tangent line to $y = \sin(\pi x)$ at $x = 2$.

    $y(2) = \sin(2\pi) = 0$. Slope $y'(2) = \pi\cos(2\pi) = \pi$. Line: $\boxed{y = \pi(x-2)}$.

133. Give an equation for the tangent line to $y = \sin(x)$ at $x = \frac{\pi}{3}$. $\boxed{y = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}(x - \dfrac{\pi}{3})}$

134. Graph the curve $y = \sqrt{x}$ and the line tangent to that curve at $(1, 1)$.

135. (a) Give the linear approximation to $\sqrt{x}$ near $x = 1$.

    $\boxed{L(x) = 1 + \frac{1}{2}(x-1)}$ (The slope is from **Task 105**, or really **Task 50(b)**.)

    (b) Use the approximation from part (a) to estimate $\sqrt{1.2}$.
    $L(1.2) = 1 + \frac{1}{2}(1.2 - 1) = 1 + \frac{0.2}{2} = \boxed{1.1}$

    (c) The true value of $\sqrt{1.2}$ is 1.09545..., so is $L(1.2)$ a good approximation?
    This question asks for an opinion, so you could answer "yes" or "no".
    I (Adam) would say "yes" because is correct to three decimal places (1.10), and the percentage error is only $\frac{1.1 - 1.09545}{1.09545} = 0.004 = 0.4\%$.

    (d) Use the approximation from part (a) to estimate $\sqrt{8}$.

    $L(5) = 1 + \frac{1}{2}(8-1) = \boxed{\frac{9}{2} = 4.5}$

    (e) The true value of $\sqrt{8}$ is 2.82843..., so is $L(8)$ a good approximation?
    I would say "no". The input 8 is not close to $x = 1$, so it is not surprising that $L(8)$ is not close to $\sqrt{8}$.

136. If f is a function with $f(-4) = 2$ and $f'(-4) = \frac{1}{3}$, give the linear approximation to $f(x)$ near $x = -4$. $\boxed{L(x) = 2 + \frac{1}{3}(x+4)}$

137. If g is a function with $g(5) = 12$ and $g'(5) = 2$, use a linear approximation to estimate the value of $g(4.9)$. $L(4.9) = \boxed{11.8}$

138. Give an equation for the tangent line to $y = \sin(x)$ at $x = \frac{\pi}{3}$. $\boxed{y = \frac{\sqrt{3}}{2} + \frac{1}{2}(x - \frac{\pi}{3})}$

☆139. Find a line that is tangent to both $y = x^2 + 20$ and $y = x^3$. $\boxed{y = 12x - 16}$ is tangent to $y = x^2 + 20$ at $x = 6$ and tangent to $y = x^3$ at $x = 2$.

140. (a) For what value(s) of x does $x^3 - 18x^2 = 0$? $\boxed{x = 0, x = 18}$

(b) For what value(s) of x does $3x^2 - 36x = 0$? $\boxed{x = 0, x = 12}$

(c) For what value(s) of x does $6x - 36 = 0$? $\boxed{x = 6}$

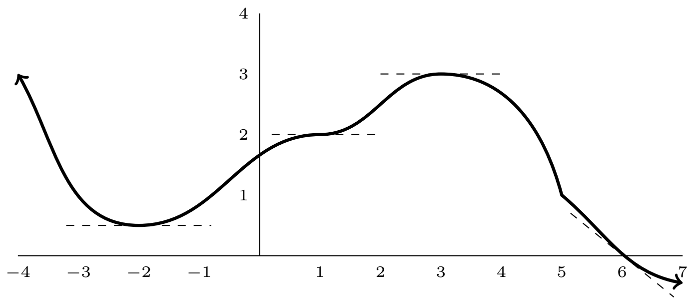
A number c is a **critical point** of $f(x)$ if either $f'(c)$ does not exist or $f'(c) = 0$.

If $f'(a) > 0$ then f is **increasing** at $x = a$.
If $f'(a) < 0$ then f is **decreasing** at $x = a$.

141. What are the critical points of $x^3 - 18x^2$? $\boxed{x = 0, x = 12}$

142. Find all the critical points of $8x^5 - 57x^4 - 24x^3 + 9$. $\boxed{0, 6, \frac{-3}{10}}$

143. List all the critical points of the function graphed below (portions of its tangent lines at $x = -2$, $x = 1$, $x = 3$, and $x = 6$ are shown as dashed lines).



Critical points are $\boxed{2, 1, 4, 5}$.

144. Is the function

$$f(x) = x^8 - 6x^3 + 29x - 12$$

increasing, decreasing, or neither when $x = -1$? $\boxed{\text{increasing}}$

145. (a) On what (possibly infinite) interval or intervals is $2x^3 - 3x^2 - 12x$ decreasing?

$\boxed{-1 < x < 2}$, which is $\boxed{(-1, 2)}$ in interval notation.

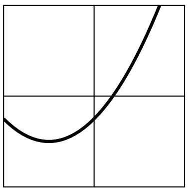
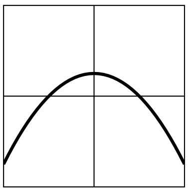
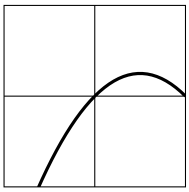
(b) On what (possibly infinite) interval or intervals is $2x^3 - 3x^2 - 12x$ increasing?

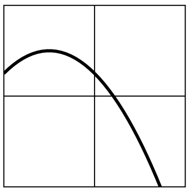
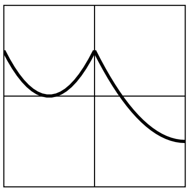
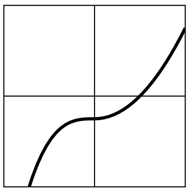
$\boxed{x < -1 \text{ or } x > 2}$, which is $\boxed{(-\infty, -1) \cup (2, \infty)}$ in interval notation.

146. List all critical points of $f(x) = \frac{3}{4}x^4 - 7x^3 + 15x^2$ in the interval $[-3, 3]$.

$f'(c) = 0$ for $c = 0, 2, 5$, but only $\boxed{0 \text{ and } 2}$ are in the interval $[-3, 3]$.

147. For each graph below, is there a critical point at $x = 0$?

(a)  (b)  (c) 

(d)  (e)  (f) 

$\boxed{\text{(a) No, (b) Yes, (c) No, (d) No, (e) Yes, (f) Yes}}$

148. The derivative of

$$f(x) = \frac{4x+1}{3x^2-12} \qquad \text{is} \qquad f'(x) = \frac{-4x^2-2x-16}{3x^4-24x^2+48}.$$

Using this, find all the critical points of $f(x)$.

$-4x^2 - 2x - 16 = 0$ has no real solutions, but $3x^4 - 24x^2 + 48 = 0$ when $\boxed{x = 2,\ x = -2}$, so f' does not exist at those points.

149. Find all the critical points of

(a) $f(x) = x^2 - \cos(x)$. $f' = 2x + \sin(x) = 0$ means $\sin(x) = -2x$, which is true only for $\boxed{x = 0}$.

(b) $f(x) = x + 2\cos(x)$. $f' = 1 - 2\sin(x) = 0$ means $\sin(x) = \frac{1}{2}$, which is true for $\boxed{x = \frac{1}{6}\pi + 2k\pi \text{ and } x = \frac{5}{6}\pi + 2k\pi}$, where k can be any integer.

(c) $f(x) = 2x + \cos(x)$. $f' = 2 - \sin(x) = 0$ when $\sin(x) = 2$, but this never happens for real values of x. So this function has $\boxed{\text{no critical points}}$.

(d) $f(x) = x^2 + x - \sin(x)$. $f' = 2x + 1 - \cos(x) = 0$ when the curves $y = \cos(x)$ and $y = 2x + 1$ intersect. This happens only at $\boxed{x = 0}$.

☆(e) $f(x) = x^2 + x + \cos(x)$. $f' = 2x + 1 - \sin(x) = 0$ when the curves $y = \sin(x)$ and $y = 2x+1$ intersect. There is one point where this occurs, but there is no nice (technically, "closed form") formula for this value. It is approximately $\boxed{x = -0.335418}$.

To find the absolute extremes of a fn. on a closed, bounded interval:
① Find the critical points of f but *ignore critical points outside the interval.*
② Compute the value of f at the critical points *and* the endpoints of the interval.
③ The point(s) from ② with the largest f-value are absolute max, and point(s) with the smallest (i.e., most negative) f-value are absolute min.

150. On the interval $[-6, 3]$, find the absolute extremes of

$$2x^3 - 21x^2 + 60x - 20.$$

$f' = 6x^2 - 42x + 60$, so CP at 2 and 5. Ignore $x = 5$ bc it's not in $[-6, 3]$. Endpoints at -6 and 3.

x	f	
-6	-1568	abs. minimum
2	32	abs. maximum
3	25	(neither)

151. Find the absolute extremes of

$$x^4 - 4x^3 + 4x^2 - 14$$

on the interval $[-3, 3]$.

$\boxed{\text{min } (f = -14) \text{ at } x = 0 \text{ and } x = 2, \text{ max } (f = 661) \text{ at } x = -3}$ Note that this function has two absolute minima (the same minimum f-value at two x-values).

152. Find the absolute extremes of $x + 2\cos(x)$ with $0 \le x \le 2\pi$.

See **Task 149(b)** for the critical points. $x = \frac{\pi}{6}$ is the only CP in the interval $[0, 2\pi]$. Including the endpoints, we have

x	f	
0	2	abs. minimum
$\pi/6$	$\frac{\pi}{6} + \sqrt{3} \approx 2.26$	(neither)
2π	$2 + 2\pi \approx 8.28$	abs. maximum

153. Find the absolute minimum and absolute maximum of

$$f(x) = \tfrac{3}{4}x^4 - 7x^3 + 15x^2$$

with $|x| \le 3$.

See **Task 146** for the CP: 0 and 2.

x	f	
-3	384.75	abs. maximum
0	0	abs. maximum
2	16	(neither)
3	6.75	(neither)

154. (a) Does the function $\dfrac{x-5}{x+2}$ have an absolute maximum on the interval $[-8, 4]$?

$\boxed{\text{no}}$ (Note: the EVT does not apply because the function is not continuous (in fact, not defined) at $x = -2$.)

(b) Does the function $\dfrac{x-5}{\cos(x)+2}$ have an absolute maximum on $[-8, 4]$?

$\boxed{\text{yes}}$ by the Extreme Value Theorem

155. A car drives in a straight line for 10 hours with its position after t hours being $24t^2 - 2t^3$ kilometers from its initial position. How far away is the farthest point the car reaches in 10 hours, and when does this occur? $\boxed{512 \text{ km away at } t = 8 \text{ hr}}$

156. (a) Calculate the derivative of $5x^2 - 3\sin(x)$. $\boxed{10x - 3\cos(x)}$

(b) Calculate the derivative of $10x - 3\cos(x)$. $\boxed{10 + 3\sin(x)}$

(c) Calculate the derivative of $10 + 3\sin(x)$. $\boxed{3\cos(x)}$

(d) Calculate the derivative of $3\cos(x)$. $\boxed{-3\sin(x)}$

(e) Calculate the derivative of $-3\sin(x)$. $\boxed{-3\cos(x)}$

The **second derivative** of a function is the derivative of its derivative. The second derivative of $y = f(x)$ with respect to x can be written as any of

$$f''(x), \qquad f'', \qquad (f')', \qquad f^{(2)}, \qquad y'', \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right], \qquad \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}, \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}.$$

We say f is **twice-differentiable** if f'' exists on the entire domain of f. Higher derivatives (third, fourth, etc.) are defined and written similarly.

A twice-differentiable function $f(x)$ is **concave up** at $x = a$ if $f''(a) > 0$.
A twice-differentiable function $f(x)$ is **concave down** at $x = a$ if $f''(a) < 0$.

An **inflection point** is a point where the concavity of a function changes.

157. Compute the following second derivatives:

(a) $f''(x)$ for $f(x) = x^{12}$ $\boxed{132x^{10}}$

(b) $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$ for $f(x) = x^3 + x^8$ $\boxed{56x^6 + 6x}$

(c) $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ for $y = 8x - 4$ $\boxed{0}$

(d) $\frac{\mathrm{d}^2}{\mathrm{d}x^2}(5x^2 - 7x + 28)$ $\boxed{10}$

(e) $f''(x)$ for $f(x) = -2x^8 + x^6 - x^3$ $\boxed{-112x^6 + 30x^4 - 6x}$

(f) $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$ for $f(x) = ax^2 + bx + c$ $\boxed{2a}$

158. Find $f'''(x) = \frac{\mathrm{d}^3 f}{\mathrm{d}x^3} = f^{(3)}(x)$ (the third derivative) for $f(x) = x^7$. $\boxed{210x^4}$

159. Give $f^{(5)}(x) = \frac{\mathrm{d}^5 f}{\mathrm{d}x^5}$ (the fifth derivative) for $f(x) = 5x^2 - 3\sin(x)$. This is **Task 156(e)**. Answer: $\boxed{-3\cos(x)}$.

160. (a) Is the function $3x^2 + 8\cos(x)$ concave up or concave down at $x = 0$? $\boxed{\text{concave down}}$

(b) Is the function $3x^2 + 5\cos(x)$ concave up or concave down at $x = 0$? $\boxed{\text{concave up}}$

161. On what interval(s) is $54x^2 - x^4$ concave up? $\boxed{-3 < x < 3}$

162. For $f(x) = x^3 - x^2 - x$,

(a) At what x value(s) does $f(x)$ change sign? That is, list values r where either $f(x) < 0$ when x is slightly less than r and $f(x) > 0$ when x is slightly more than r, or $f(x) > 0$ when x is slightly less than r and $f(x) < 0$ when x is slightly more than r.

$\boxed{x = \frac{1-\sqrt{5}}{2}, x = 0, x = \frac{1+\sqrt{5}}{2}}$

(b) At what x value(s) does $f'(x)$ change sign? $\boxed{x = \frac{-1}{3}, x = 1}$

(c) At what x value(s) does $f''(x)$ change sign? $\boxed{x = \frac{1}{3}}$

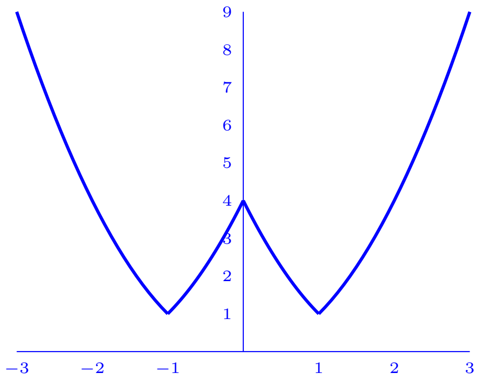
(d) List all inflection points of $x^3 - x^2 - x$. same as (c): $\boxed{x = \frac{1}{3}}$

☆163. Give an example of a function with one local maximum and two local minimums but no inflection points.

In order to avoid inflection points, the function must have the same concavity everywhere. An example that is concave up everywhere is

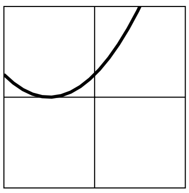
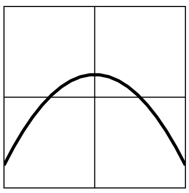
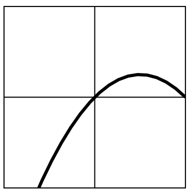
$$f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ (x+2)^2 & \text{if } -1 \le x < 0 \\ (x-2)^2 & \text{if } 0 \le x < 1 \\ x^2 & \text{if } x \ge 1. \end{cases}$$

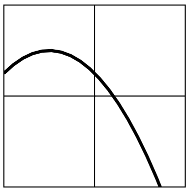
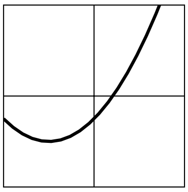
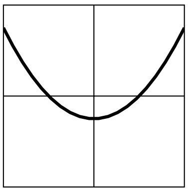
The graph of this is



A similar-looking example that is neither concave up nor concave down is $g(x) = |x+1| + |x-1| - |x|$.

164. Which graph below has $f'(0) = 1$ and $f''(0) = -1$? $\boxed{\text{C}}$

(A)  (B)  (C) 

(D)  (E)  (F) 

For a twice-differentiable function $f(x)$ with a critical point at $x = c$, ...

The Second Derivative Test:
- If $f''(c) > 0$ then f has a local minimum at $x = c$.
- If $f''(c) < 0$ then f has a local maximum at $x = c$.
- If $f''(c) = 0$ the test is inconclusive.

The First Derivative Test:
- If $f'(x) < 0$ to the left of $x = c$ and $f'(x) > 0$ to the right of $x = c$ then f has a local minimum at $x = c$.
- If $f'(x) > 0$ to the left of $x = c$ and $f'(x) < 0$ to the right of $x = c$ then f has a local maxium at $x = c$.
- If $f'(x)$ has the same sign on both sides of $x = c$ then $x = c$ is neither a local minimum nor a local maximum.

165. Find all critical points of

$$4x^3 + 21x^2 - 24x + 19$$

and classify each as a local minimum, local maximum, or neither.

local max at $x = -4$, local min at $x = 1/2$

166. Find and classify the critical points of $f(x) = x^4 - 4x^3 - 36x^2 + 18$.

$x = -3$ is local min. $x = 0$ is local max. $x = 6$ is local min.

167. Find the inflection points of the function from Task 166. $1 - \sqrt{7},\ 1 + \sqrt{7}$

☆168. Find and classify the critical points of $f(x) = x(6 - x)^{2/3}$.

After simplifying, $f'(x) = \dfrac{18 - 5x}{3(6 - x)^{1/3}}$, so the critical points are $x = \frac{18}{5}$ (where f' is zero) and $x = 6$ (where f' doesn't exist). The fact that $x = \frac{18}{5}$ is a local max can be found from the First or the Second Derivative Test, but the fact that $x = 6$ is a local min requires the First D. Test because $f''(6)$ is not defined.

169. Find and classify the critical points of

$$\tfrac{3}{2}x^4 - 16x^3 + 63x^2 - 108x + 51.$$

$x = 2$ is local min. $x = 3$ is *neither*.

170. Label each of following statements as true or false:

(a) "Every critical point of a differentiable function is also a local minimum."
false

(b) “Every local minimum of a differentiable function is also a critical point.” $\boxed{\text{true}}$

(c) “Every critical point of a differentiable function is also an inflection point.” $\boxed{\text{false}}$

(d) “Every inflection point of a differentiable function is also a critical point.” $\boxed{\text{false}}$

171. For the function

$$f(x) = \tfrac{1}{8}x^4 - 3x^2 + 8x + 15,$$

find

(a) the interval(s) where f is monotonically increasing, $\boxed{(-4, 2) \cup (2, \infty)}$

(b) the interval(s) where f is monotonically decreasing, $\boxed{(-\infty, -4)}$

(c) the critical points, $\boxed{x = -4, x = 2}$

(d) all local minima, $\boxed{x = -4}$, at which point $f = -33$

(e) all local maxima, $\boxed{\text{(none)}}$

(f) and the inflection points. $\boxed{x = -2, x = 2}$

☆172. What is the maximum number of inflection points that a function of the form

$$\_\_x^6 + \_\_x^5 + \_\_x^4 + \_\_x^3 + \_\_x^2 + \_\_x + \_\_$$

can have? $\boxed{4}$ because f'' will be a degree-4 polynomial.

☆173. Give two critical points of $\sin(5^{\cos(2x^3+8)})$. The derivative is

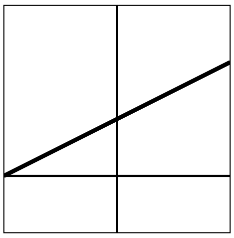
$$f'(x) = \cos(5^{\cos(2x^3+8)}) \cdot 5^{\cos(2x^3+8)} \ln(5) \cdot \big(- \sin(2x^3 + 8)\big) \cdot 6x^2,$$

and we want $f'(x) = 0$. The simplest answer is $\boxed{x = 0}$ because

$$f'(0) = (\dots) \cdot 5(0)^2 = 0.$$

Since $f' = (\dots) \cdot \sin(2x^3 + 8) \cdot (\dots)$, we also get a critical point any time $\sin(2x^3 + 8) = 0$. One way to do this is by having $2x^3 + 8 = 0$, which leads to $\boxed{x = -\sqrt[3]{4}}$. (There are infinitely many other answers, such as $x = \sqrt[3]{\frac{\pi-8}{2}}$.)

174. Match the functions (a)-(f) to their derivatives (I)-(VI).

(a)


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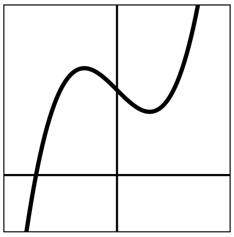

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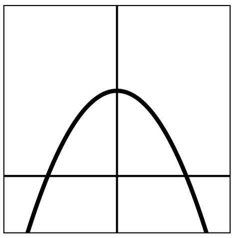

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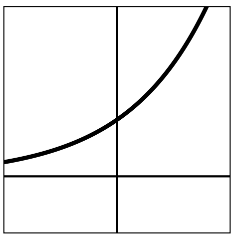

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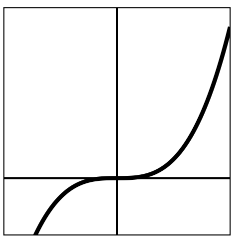

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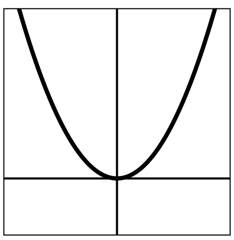
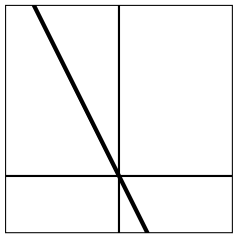
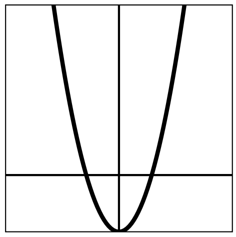


(I)


(II)


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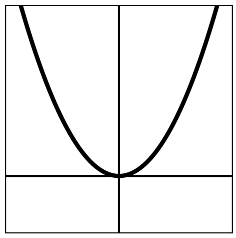

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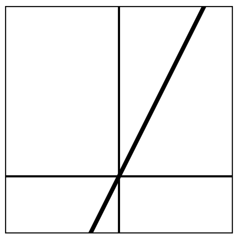
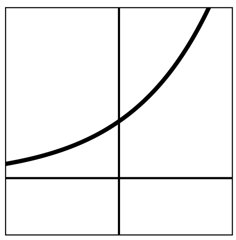
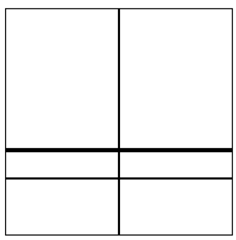


(V)


(VI)


$\boxed{\text{a-VI, b-II, c-I, d-V, e-III, f-IV}}$