Math for Management, Winter 2023

# List 6

*Derivative applications (monotonicity, convexity, min/max)*

128. Calculate $f'(2)$ for the function $f(x) = x^4 + 4x$.

129. Find the *slope* of the tangent line to $y = x^4 + 4x$ at the point $(2, 24)$.

130. Give an *equation* for the tangent line to $y = x^4 + 4x$ through the point $(2, 24)$.

131. Give an equation for the tangent line to $y = \dfrac{1}{\sqrt{x}}$ at $x = 4$.

132. Give an equation for the tangent line to $y = \sin(\pi x)$ at $x = 2$.

133. Give an equation for the tangent line to $y = \sin(x)$ at $x = \frac{\pi}{3}$.

134. Graph the curve $y = \sqrt{x}$ and the line tangent to that curve at $(1, 1)$.

135. (a) Give the linear approximation to $\sqrt{x}$ near $x = 1$.
   (b) Use the approximation from part (a) to estimate $\sqrt{1.2}$.
   (c) The true value of $\sqrt{1.2}$ is 1.09545..., so is $L(1.2)$ a good approximation?
   (d) Use the approximation from part (a) to estimate $\sqrt{8}$.
   (e) The true value of $\sqrt{8}$ is 2.82843..., so is $L(8)$ a good approximation?

136. If $f$ is a function with $f(-4) = 2$ and $f'(-4) = \frac{1}{3}$, give the linear approximation to $f(x)$ near $x = -4$.

137. If $g$ is a function with $g(5) = 12$ and $g'(5) = 2$, use a linear approximation to estimate the value of $g(4.9)$.

138. Give an equation for the tangent line to $y = \sin(x)$ at $x = \frac{\pi}{3}$.

☆139. Find a line that is tangent to both $y = x^2 + 20$ and $y = x^3$.

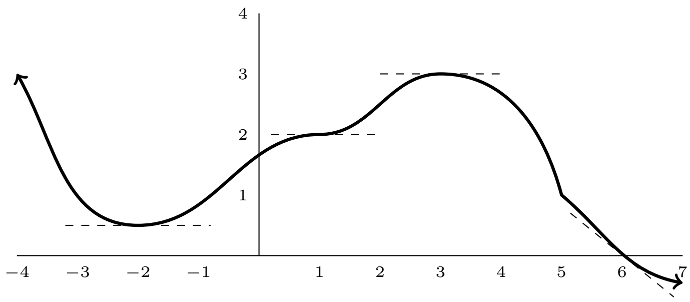
140. (a) For what value(s) of $x$ does $x^3 - 18x^2 = 0$?
   (b) For what value(s) of $x$ does $3x^2 - 36x = 0$?
   (c) For what value(s) of $x$ does $6x - 36 = 0$?

> A number $c$ is a **critical point** of $f(x)$ if either $f'(c)$ does not exist or $f'(c) = 0$.
>
> If $f'(a) > 0$ then $f$ is **increasing** at $x = a$.
> If $f'(a) < 0$ then $f$ is **decreasing** at $x = a$.

141. What are the critical points of $x^3 - 18x^2$?

142. Find all the critical points of $8x^5 - 57x^4 - 24x^3 + 9$.

143. List all the critical points of the function graphed below (portions of its tangent lines at $x = -2$, $x = 1$, $x = 3$, and $x = 6$ are shown as dashed lines).




144. Is the function

$$f(x) = x^8 - 6x^3 + 29x - 12$$

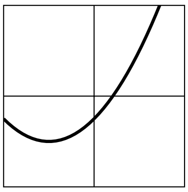
increasing, decreasing, or neither when $x = -1$?

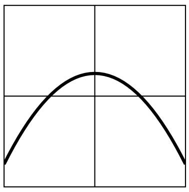
145. (a) On what (possibly infinite) interval or intervals is $2x^3 - 3x^2 - 12x$ decreasing?

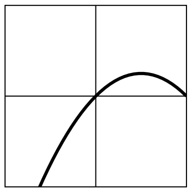
(b) On what (possibly infinite) interval or intervals is $2x^3 - 3x^2 - 12x$ increasing?

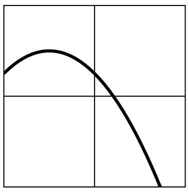
146. List all critical points of $f(x) = \frac{3}{4}x^4 - 7x^3 + 15x^2$ in the interval $[-3, 3]$.

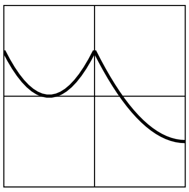
147. For each graph below, is there a critical point at $x = 0$?

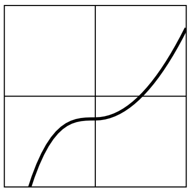
(a) 

(b) 

(c) 

(d) 

(e) 

(f) 

148. The derivative of

$$f(x) = \frac{4x+1}{3x^2 - 12} \qquad \text{is} \qquad f'(x) = \frac{-4x^2 - 2x - 16}{3x^4 - 24x^2 + 48}.$$

Using this, find all the critical points of $f(x)$.

149. Find all the critical points of

(a) $f(x) = x^2 - \cos(x)$.

(b) $f(x) = x + 2\cos(x)$.

(c) $f(x) = 2x + \cos(x)$.

(d) $f(x) = x^2 + x - \sin(x)$.

☆(e) $f(x) = x^2 + x + \cos(x)$.

**To find the absolute extremes of a fn. on a closed, bounded interval:**
① Find the critical points of $f$ but *ignore critical points outside the interval.*
② Compute the value of $f$ at the critical points *and* the endpoints of the interval.
③ The point(s) from ② with the largest $f$-value are absolute max, and point(s) with the smallest (i.e., most negative) $f$-value are absolute min.

150. On the interval $[-6, 3]$, find the absolute extremes of

$$2x^3 - 21x^2 + 60x - 20.$$

151. Find the absolute extremes of

$$x^4 - 4x^3 + 4x^2 - 14$$

on the interval $[-3, 3]$.

152. Find the absolute extremes of $x + 2\cos(x)$ with $0 \le x \le 2\pi$.

153. Find the absolute minimum and absolute maximum of

$$f(x) = \tfrac{3}{4}x^4 - 7x^3 + 15x^2$$

with $|x| \le 3$.

154. (a) Does the function $\dfrac{x-5}{x+2}$ have an absolute maximum on the interval $[-8, 4]$?

(b) Does the function $\dfrac{x-5}{\cos(x)+2}$ have an absolute maximum on $[-8, 4]$?

155. A car drives in a straight line for 10 hours with its position after $t$ hours being $24t^2 - 2t^3$ kilometers from its initial position. How far away is the farthest point the car reaches in 10 hours, and when does this occur?

156. (a) Calculate the derivative of $5x^2 - 3\sin(x)$.

(b) Calculate the derivative of $10x - 3\cos(x)$.

(c) Calculate the derivative of $10 + 3\sin(x)$.

(d) Calculate the derivative of $3\cos(x)$.

(e) Calculate the derivative of $-3\sin(x)$.

The **second derivative** of a function is the derivative of its derivative. The second derivative of $y = f(x)$ with respect to $x$ can be written as any of

$$f''(x), \quad f'', \quad (f')', \quad f^{(2)}, \quad y'', \quad \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right], \quad \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}, \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}.$$

We say $f$ is **twice-differentiable** if $f''$ exists on the entire domain of $f$. Higher derivatives (third, fourth, etc.) are defined and written similarly.

A twice-differentiable function $f(x)$ is **concave up** at $x = a$ if $f''(a) > 0$.
A twice-differentiable function $f(x)$ is **concave down** at $x = a$ if $f''(a) < 0$.

An **inflection point** is a point where the concavity of a function changes.

157. Compute the following second derivatives:

(a) $f''(x)$ for $f(x) = x^{12}$

(b) $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$ for $f(x) = x^3 + x^8$

(c) $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ for $y = 8x - 4$

(d) $\frac{\mathrm{d}^2}{\mathrm{d}x^2}(5x^2 - 7x + 28)$

(e) $f''(x)$ for $f(x) = -2x^8 + x^6 - x^3$

(f) $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$ for $f(x) = ax^2 + bx + c$

158. Find $f'''(x) = \frac{\mathrm{d}^3 f}{\mathrm{d}x^3} = f^{(3)}(x)$ (the third derivative) for $f(x) = x^7$.

159. Give $f^{(5)}(x) = \frac{\mathrm{d}^5 f}{\mathrm{d}x^5}$ (the fifth derivative) for $f(x) = 5x^2 - 3\sin(x)$.

160. (a) Is the function $3x^2 + 8\cos(x)$ concave up or concave down at $x = 0$?

(b) Is the function $3x^2 + 5\cos(x)$ concave up or concave down at $x = 0$?

161. On what interval(s) is $54x^2 - x^4$ concave up?

162. For $f(x) = x^3 - x^2 - x$,

(a) At what $x$ value(s) does $f(x)$ change sign? That is, list values $r$ where either $f(x) < 0$ when $x$ is slightly less than $r$ and $f(x) > 0$ when $x$ is slightly more than $r$, or $f(x) > 0$ when $x$ is slightly less than $r$ and $f(x) < 0$ when $x$ is slightly more than $r$.

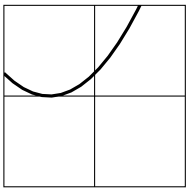
(b) At what $x$ value(s) does $f'(x)$ change sign?

(c) At what $x$ value(s) does $f''(x)$ change sign?

(d) List all inflection points of $x^3 - x^2 - x$.

☆163. Give an example of a function with one local maximum and two local minimums but no inflection points.

164. Which graph below has $f'(0) = 1$ and $f''(0) = -1$?

(A)

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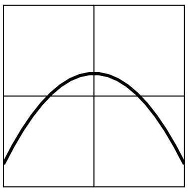
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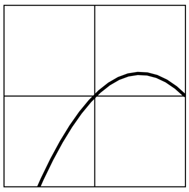
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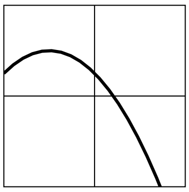
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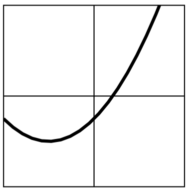
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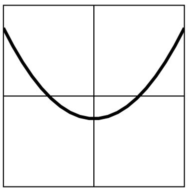


For a twice-differentiable function $f(x)$ with a critical point at $x = c$, ...

**The Second Derivative Test:**
- If $f''(c) > 0$ then $f$ has a local minimum at $x = c$.
- If $f''(c) < 0$ then $f$ has a local maximum at $x = c$.
- If $f''(c) = 0$ the test is inconclusive.

**The First Derivative Test:**
- If $f'(x) < 0$ to the left of $x = c$ and $f'(x) > 0$ to the right of $x = c$ then $f$ has a local minimum at $x = c$.
- If $f'(x) > 0$ to the left of $x = c$ and $f'(x) < 0$ to the right of $x = c$ then $f$ has a local maxium at $x = c$.
- If $f'(x)$ has the same sign on both sides of $x = c$ then $x = c$ is neither a local minimum nor a local maximum.

165. Find all critical points of

$$4x^3 + 21x^2 - 24x + 19$$

and classify each as a local minimum, local maximum, or neither.

166. Find and classify[1] the critical points of

$$f(x) = x^4 - 4x^3 - 36x^2 + 18.$$

167. Find the inflection points of the function from Task 166.

☆168. Find and classify the critical points of $f(x) = x(6-x)^{2/3}$.

169. Find and classify the critical points of

$$\tfrac{3}{2}x^4 - 16x^3 + 63x^2 - 108x + 51.$$

170. Label each of following statements as true or false:
   (a) "Every critical point of a differentiable function is also a local minimum."
   (b) "Every local minimum of a differentiable function is also a critical point."
   (c) "Every critical point of a differentiable function is also an inflection point."
   (d) "Every inflection point of a differentiable function is also a critical point."

171. For the function

$$f(x) = \tfrac{1}{8}x^4 - 3x^2 + 8x + 15,$$

find
   (a) the interval(s) where $f$ is monotonically increasing,
   (b) the interval(s) where $f$ is monotonically decreasing,
   (c) the critical points,
   (d) all local minima,
   (e) all local maxima,
   (f) and the inflection points.

☆172. What is the maximum number of inflection points that a function of the form

$$\_\_x^6 + \_\_x^5 + \_\_x^4 + \_\_x^3 + \_\_x^2 + \_\_x + \_\_$$

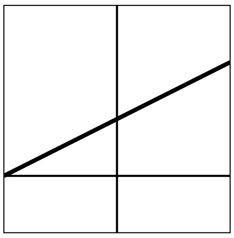
can have?

☆173. Give two critical points of $\sin(5^{\cos(2x^3+8)})$.

[1]"Classify the critical points" means to say whether each one is a local minimum, local maximum, or neither.

174. Match the functions (a)-(f) to their derivatives (I)-(VI).

(a)



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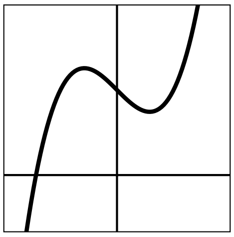

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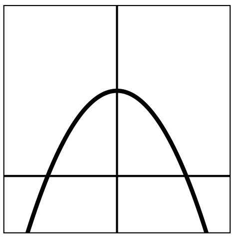

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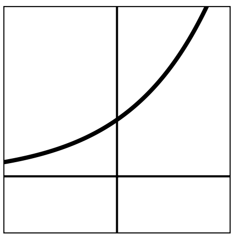

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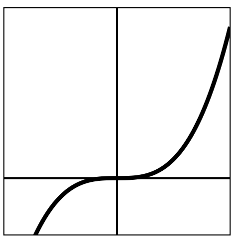
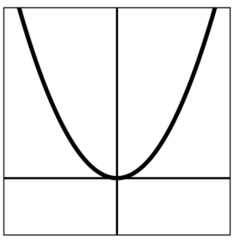
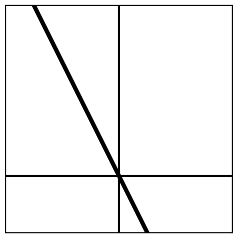


(f)



(I)



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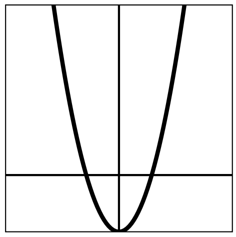

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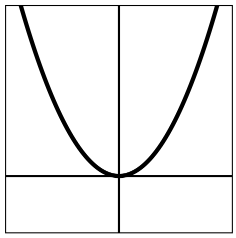

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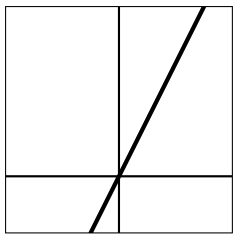

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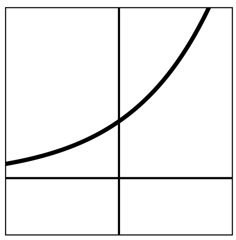

(VI)