Math for Management, Winter 2023

List 7

Integrals (definite, indefinite, u-sub., parts)

An **indefinite integral** describes all the anti-derivatives of a function. We write

$$\int f(x)\,dx = F(x) + C,$$

where $F(x)$ is any function for which $F'(x) = f(x)$.

175. Find $\int (6x^5 + 7x^3 - 9)\,\mathrm{d}x = \boxed{x^6 + \frac{7}{4}x^4 - 9x + C}$

176. Find $\int (6u^5 + 7u^3 - 9)\,\mathrm{d}u = \boxed{u^6 + \frac{7}{4}u^4 - 9u + C}$

177. Give each of the following indefinite integrals using basic derivative knowledge:

   (a) $\int x^{372.5}\,\mathrm{d}x = \boxed{\frac{1}{373.5}x^{373.5} + C}$

   (b) $\int \frac{1}{x}\,\mathrm{d}x = \boxed{\ln(x) + C}$

   (c) $\int e^x\,\mathrm{d}x = \boxed{e^x + C}$

   (d) $\int 97^x\,\mathrm{d}x = \boxed{\frac{1}{\ln(97)}97^x + C}$

   (e) $\int -\sin(x)\,\mathrm{d}x = \boxed{\cos(x) + C}$

   (f) $\int \sin(x)\,\mathrm{d}x = \boxed{-\cos(x) + C}$

   (g) $\int \cos(x)\,\mathrm{d}x = \boxed{\sin(x) + C}$

   (h) $\int 5t^9\,\mathrm{d}t = \boxed{\frac{1}{2}t^{10} + C}$

178. If $u = 6x^2 - 5$, give a formula for $\mathrm{d}u$ (this formula will have x and $\mathrm{d}x$ in it) and a formula for $\mathrm{d}x$ (this formula will have x and $\mathrm{d}u$ in it).

   $\boxed{\mathrm{d}u = 12x\,\mathrm{d}x}$ and $\boxed{\mathrm{d}x = \frac{\mathrm{d}u}{12x}}$

Substitution: $\int f(u(x)) \cdot u'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$

179. (a) Re-write $\int \frac{x}{(6x^2-5)^3}\,\mathrm{d}x$ as $\int \ldots\,\mathrm{d}u$ using the substitution $u = 6x^2 - 5$.

   $\mathrm{d}u = 12x\,\mathrm{d}x$, so $x\,\mathrm{d}x = \frac{1}{12}\,\mathrm{d}u$ and the integral is $\boxed{\int \frac{1}{12u^3}\,\mathrm{d}u}$.

   (b) Find $\int \frac{x}{(6x^2-5)^3}\,\mathrm{d}x = \boxed{\frac{-1}{24(6x^2-5)^2} + C}$

180. (a) Re-write $\int x^3\sin(x^4)\,\mathrm{d}x$ as $\int \ldots \mathrm{d}u$ using the substitution $u=x^4$. $\boxed{\int \frac{1}{4}\sin(u)\,\mathrm{d}u}$

(b) Find $\int x^3\sin(x^4)\,\mathrm{d}x = \boxed{-\frac{1}{4}\cos(x^4)+C}$

181. (a) Re-write $\int x\sin(x^4)\,\mathrm{d}x$ as $\int \ldots \mathrm{d}u$ using the substitution $u=x^2$. $\boxed{\int \frac{1}{2}\sin(u^2)\,\mathrm{d}u}$

☆(b) Find $\int x\sin(x^4)\,\mathrm{d}x$ There is literally no "elementary" formula for this. You might see $\sqrt{\pi/8}S(x^2\sqrt{2/\pi})$ in some sources, but this is just re-writing the integral using a special short-hand for this "Fresnel" integral.

182. Give $\int \frac{2x^3-7x+3}{5x^4-35x^2+30x+125}\,\mathrm{d}x = \boxed{\frac{1}{10}\ln(x^4-7x^2+6x+25)+C}$

183. Find $\int \cot(x)\,\mathrm{d}x$ using substitution. Hint: $\cot(x)=\frac{\cos(x)}{\sin(x)}$.

With $u=\sin(x)$, we have $\mathrm{d}u=\cos(x)\,\mathrm{d}x$ and

$$\int \frac{\cos(x)\,\mathrm{d}x}{\sin(x)} = \int \frac{\mathrm{d}u}{u} = \ln|u|+C = \boxed{\ln|\sin(x)|+C}$$

Technically the "C" could actually be a piecewise function that is constant on each interval where $\ln|\sin(x)|$ is continuous. But it is common to just write "$+C$" anyway.

The notation $\boldsymbol{g(x)}\Big|_{\boldsymbol{x=a}}^{\boldsymbol{x=b}}$ or $g\Big|_a^b$ or $\Big[g(x)\Big]_a^b$ means to do the subtraction $g(b)-g(a)$.

184. Calculate $\frac{1}{3}x^3\Big|_{x=1}^{x=2}$. $\frac{1}{3}(2)^3-\frac{1}{3}(1)^3=\frac{8}{3}-\frac{1}{3}=\boxed{\frac{7}{3}}$

185. Calculate $(x^3+\frac{1}{2}x)\Big|_{x=1}^{x=5} = \Big[x^3+\frac{1}{2}x\Big]_{x=1}^{x=5}$. $\boxed{126}$

186. Calculate $\frac{1-x}{e^x}\Big|_{x=0}^{x=1}$. $\boxed{-1}$

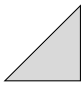
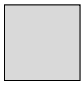
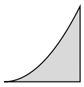
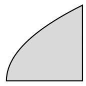
The **definite integral** $\int_a^b f(x)\,dx$, spoken as "the integral from a to b of $f(x)$ with respect to x", is the (signed) area of the region with $x=a$ on the left, $x=b$ on the right, $y=0$ at the bottom, and $y=f(x)$ at the top (but if $f(x)<0$ for some x or if $b<a$ then it's possible for the area to be negative).

The **Newton–Leibniz Theorem** (**NL**, also called the "Fundamental Theorem of Calculus" or FTC) says that

$$\int_a^b f(x)\,\mathrm{d}x = F(x)\Big|_{x=a}^{x=b} = F(b)-F(a),$$

where $F(x)$ is any function for which $F'(x)=f(x)$.

187. Match the shapes (a)-(d) with the integral (I)-(IV) that is most likely to calculate its area.

(a)  (b)  (c)  (d)  a-II, b-IV, c-III, d-I

(I) $\int_0^1 \sqrt{x}\,\mathrm{d}x$ (II) $\int_0^1 x\,\mathrm{d}x$ (III) $\int_0^1 x^2\,\mathrm{d}x$ (IV) $\int_0^1 1\,\mathrm{d}x$

188. Calculate $\int^2 x^2\,\mathrm{d}x$ using the Newton–Leibniz Theorem. This is exactly **Task 184**.

Answer: $\boxed{\frac{7}{3}}$

189. Write, in symbols, the integral from zero to six of x^2 with respect to x, then find the value of that definite integral. $\boxed{\int_0^6 x^2\,\mathrm{d}x = 72}$

190. Evaluate (meaning find of the value of) the following definite integrals using common area formulas.

(a) $\int_3^9 2\,\mathrm{d}x = \boxed{12}$

(b) $\int_3^9 -2\,\mathrm{d}x = \boxed{-12}$

(c) $\int_0^5 x\,\mathrm{d}x = \boxed{\frac{25}{2}}$

(d) $\int_{-2}^4 |x|\,\mathrm{d}x = \boxed{10}$

(e) $\int_0^5 3x\,\mathrm{d}x = \boxed{\frac{75}{2}}$

(f) $\int_1^5 3x\,\mathrm{d}x = \boxed{36}$

(g) $\int_{-4}^4 \sqrt{16 - x^2}\,\mathrm{d}x = \boxed{8\pi}$

(h) $\int_0^7 \sqrt{49 - x^2}\,\mathrm{d}x = \boxed{\frac{49}{4}\pi}$

191. Evaluate the following definite integrals using the FTC. Your answer for each should be a number.

(a) $\int_{-3}^9 2\,\mathrm{d}x = 2x\Big|_{x=-3}^{x=9} = 18 - (-6) = \boxed{24}$

(b) $\int_1^5 3x\,\mathrm{d}x = \frac{3}{2}x^2\Big|_{x=1}^{x=5} = \frac{75}{2} - \frac{3}{2} = \boxed{36}$

(c) $\int_1^{12} \frac{1}{x}\,\mathrm{d}x = \ln(x)\Big|_{x=1}^{x=12} = \ln(12) - \ln(0) = \boxed{\ln(12)}$

(d) $\displaystyle\int_0^9 (x^3 - 9x)\,\mathrm{d}x = \left(\tfrac{1}{4}x^4 - \tfrac{9}{2}x^2\right)\Big|_{x=0}^{x=9} = \tfrac{5103}{4} - 0 = \boxed{\dfrac{5103}{4}}$

(e) $\displaystyle\int_0^\pi \sin(t)\,\mathrm{d}t = (-\cos(t))\Big|_{t=0}^{t=\pi} = -\cos(\pi) - (-\cos(0)) = \boxed{2}$

(f) $\displaystyle\int_2^8 3\sqrt{u}\,\mathrm{d}u = 2x^{3/2}\Big|_{x=2}^{x=8} = 32\sqrt{2} - 4\sqrt{2} = \boxed{28\sqrt{2}}$

(g) $\displaystyle\int_0^1 (e^x + x^e)\,\mathrm{d}x = \boxed{\dfrac{e^2}{1+e}}$

(h) $\displaystyle\int_{-1}^1 x^2\,\mathrm{d}x = \boxed{\dfrac{2}{3}}$

(i) $\displaystyle\int_1^3 t\,\mathrm{d}t = \boxed{4}$

(j) $\displaystyle\int_9^9 \sin(x^2)\,\mathrm{d}x = \boxed{0}$

(k) $\displaystyle\int_0^5 \cos(x)\,\mathrm{d}x = \boxed{\sin(5)}$

192. If $\displaystyle\int_1^4 f(x)\,\mathrm{d}x = 12$ and $\displaystyle\int_1^6 f(x)\,\mathrm{d}x = 15$, what is the value of $I = \displaystyle\int_4^6 f(x)\,\mathrm{d}x$?

$\displaystyle\int_1^6 f(x)\,\mathrm{d}x = \int_1^4 f(x)\,\mathrm{d}x + \int_4^6 f(x)\,\mathrm{d}x$, so $12 + I = 14$, and thus $I = \boxed{3}$.

193. If $\displaystyle\int_0^1 f(x)\,\mathrm{d}x = 7$ and $\displaystyle\int_0^1 g(x)\,\mathrm{d}x = 3$, calculate each of the following or say that there is not enough information to possibly do the calculation.

(a) $\displaystyle\int_0^1 \big(f(x) + g(x)\big)\,\mathrm{d}x = \boxed{10}$

(b) $\displaystyle\int_0^1 \big(f(x) - g(x)\big)\,\mathrm{d}x = \boxed{4}$

(c) $\displaystyle\int_0^1 \big(f(x) \cdot g(x)\big)\,\mathrm{d}x$ $\boxed{\text{not enough info}}$

(d) $\displaystyle\int_0^1 \big(5f(x)\big)\,\mathrm{d}x = \boxed{35}$

(e) $\displaystyle\int_0^1 \big(f(x)^5\big)\,\mathrm{d}x$ $\boxed{\text{not enough info}}$

194. Which of the following has the same value as $\displaystyle\int_2^4 \frac{3x^2 - 2}{\ln(x^3 - 2x + 1)}\,\mathrm{d}x$?

(A) $\displaystyle\int_5^{57} \frac{1}{\ln(u)}\,\mathrm{d}u$ (B) $\displaystyle\int_2^4 \frac{1}{\ln(u)}\,\mathrm{d}u$ (C) $\displaystyle\int_{10}^{46} \frac{1}{\ln(u)}\,\mathrm{d}u$ (D) $\displaystyle\int_1^2 \frac{1}{\ln(u)}\,\mathrm{d}u$

Using the substitution $u = x^3 - 2x + 1$, we have $\mathrm{d}u = (3x^2 - 2)\,\mathrm{d}x$, so

$$\frac{(3x^2-2)\,\mathrm{d}x}{\ln(x^3-2x+1)} = \frac{\mathrm{d}u}{\ln(u)}.$$

The values 2 and 4 are x-values, so *these change* when we write $\int \dots \mathrm{d}u$.

When $x = 2$, $u = 2^3 - 2(2) + 1 = 5$. When $x = 4$, $u = 4^3 - 2(4) + 1 = 57$. $\boxed{\text{(A)}}$

195. Find the following integrals using substitution:

(a) $\int (5-x)^{10}\,\mathrm{d}x = \int -u^{10}\,\mathrm{d}u = \frac{-1}{11}u^{11} + C = \boxed{\frac{-1}{11}(5-x)^{11} + C}$
using $u = 5 - x$, so $\mathrm{d}u = -\,\mathrm{d}x$.

(b) $\displaystyle\int_1^3 \frac{x}{(6x^2-5)^3}\,\mathrm{d}x$

Indefinite: $\displaystyle\int \frac{x}{(6x^2-5)^3}\,\mathrm{d}x = \int \tfrac{1}{12}u^{-3}\,\mathrm{d}u = \tfrac{-1}{24}u^{-2} + C = \frac{-1}{24(6x^2-5)^2} + C$
using $u = 6x^2 - 5$, so $\mathrm{d}u = 12x\,\mathrm{d}x$. Then the definite integral is

$$\int_1^3 \frac{x}{(6x^2-5)^3}\,\mathrm{d}x = \left.\frac{-1}{24(6x^2-5)^2}\right|_{x=1}^{x=3} = \boxed{\frac{100}{2401}}.$$

Alternatively, when $x = 1, u = 6(1)^2 - 5 = 1$ and when $x = 3, u = 6(3)^2 - 5 = 49$, so this is

$$\int_1^{49} \tfrac{1}{12}u^{-3}\,\mathrm{d}u = \left.\frac{-1}{24u^2}\right|_{u=1}^{u=49} = \boxed{\frac{100}{2401}}.$$

(c) $\int \sqrt{4x+3}\,\mathrm{d}x = \int \frac{1}{4}u^{1/2}\,\mathrm{d}u = \frac{1}{6}u^{3/2} + C = \boxed{\frac{1}{6}(4x+3)^{3/2} + C}$
using $u = 4x + 3$, so $\mathrm{d}u = 4\,\mathrm{d}x$.

(d) $\displaystyle\int_0^{\sqrt{\pi}} x\sin(x^2)\,\mathrm{d}x$

Indefinite: $\int x\sin(x^2)\,\mathrm{d}x = \int \frac{1}{2}\sin(u)\,\mathrm{d}u = \frac{-1}{2}\cos(u) + C = \frac{-1}{2}\cos(x^2) + C$
using $u = x^2$, so $\mathrm{d}u = 2x\,\mathrm{d}x$. Then the definite integral is

$$\int_0^{\sqrt{\pi}} x\sin(x^2)\,\mathrm{d}x = \left.\tfrac{-1}{2}\cos(x^2)\right|_{x=0}^{x=\sqrt{\pi}} = \tfrac{-1}{2}(-1) - \tfrac{-1}{2}(1) = \boxed{1}.$$

Alternatively, when $x = 0, u = 0$ and when $x = \sqrt{\pi}, u = \pi$, so this is

$$\int_0^{\pi} \tfrac{1}{2}\sin(u)\,\mathrm{d}u = \left.\tfrac{-1}{2}\cos(u)\right|_{u=0}^{u=\pi} = \tfrac{-1}{2}(-1) - \tfrac{-1}{2}(1) = \boxed{1}.$$

(e) $\displaystyle\int \frac{5}{4x+9}\,\mathrm{d}x = \boxed{\frac{5}{4}\ln(4x+9) + C}$

(f) $\displaystyle\int \frac{5x}{4x^2+9}\,\mathrm{d}x = \boxed{\frac{5}{8}\ln(4x^2+9) + C}$

☆(g) $\displaystyle\int \frac{5}{4x^2+9}\,dx = \boxed{\frac{5}{6}\arctan(\tfrac{2}{3}x)+C}$

(h) $\displaystyle\int \frac{\sin(\ln(x))}{x}\,dx = \int \sin(u)\,du = -\cos(u)+C = \boxed{-\cos(\ln(x))+C}$

using $u=\ln(x)$, so $du = \frac{1}{x}\,dx$.

☆(i) $\displaystyle\int_0^9 \sqrt{4-\sqrt{x}}\,dx$

$u = 4-\sqrt{x}$ gives $du = \frac{-1}{2\sqrt{x}}\,dx$. There is no "$\frac{-1}{2\sqrt{x}}$" in the original integral, but $dx = -2\sqrt{x}\,du$ and $\sqrt{x} = 4-u$, so $dx = -2(4-u)\,du = (2u-8)\,du$. Thus

$$\int \sqrt{4-\sqrt{x}}\,dx = \int \sqrt{u}\,(2u-8)du = \int (2u^{3/2}-8u^{1/2})\,du$$
$$= \tfrac{4}{5}u^{5/2} - \tfrac{16}{3}u^{3/2}+C.$$

The definite integral is

$$\left[\tfrac{4}{5}(4-x^{1/2})^{5/2} - \tfrac{16}{3}(4-x^{1/2})^{3/2}\right]_{x=0}^{x=9} = \left(\frac{-68}{15}\right) - \left(\frac{-125}{15}\right) = \boxed{\frac{188}{15}}$$

or, with $4-\sqrt{0}=4$ and $4-\sqrt{9}=1$,

$$\left[\tfrac{4}{5}u^{5/2} - \tfrac{16}{3}u^{3/2}\right]_{u=4}^{u-1} = \left(\frac{-68}{15}\right) - \left(\frac{-125}{15}\right) = \boxed{\frac{188}{15}}$$

(j) $\displaystyle\int x^3\cos(2x^4)\,dx = \int \tfrac{1}{8}\cos(u)\,du = \tfrac{1}{8}\sin(u)+C = \boxed{\tfrac{1}{8}\sin(2x^4)+C}$

using $u=2x^4$, so $du = 8x^3\,dx$.

(k) $\displaystyle\int e^{t^5}t^4\,dt = \int e^u\tfrac{1}{5}\,du = \tfrac{1}{5}e^u + C = \boxed{\tfrac{1}{5}e^{t^5}+C}$

using $u=t^5$, so $du = 5t^4\,dt$.

(l) $\displaystyle\int \frac{(\ln(x))^2}{5x}\,dx = \boxed{\frac{(\ln(x))^3}{15}}$

(m) $\displaystyle\int \frac{1}{x\ln(x)}\,dx = \int \frac{1}{u}\,du = \ln(u)+C = \boxed{\ln(\ln(x))+C}$

using $u=\ln(x)$, so $du = \frac{1}{x}\,dx$.

(n) $\displaystyle\int_0^{\pi/2} \sin(x)\cos(x)\,dx = \boxed{\frac{1}{2}}$

(o) $\displaystyle\int \sin(1-x)(2-\cos(1-x))^4\,dx = \boxed{\frac{-1}{5}(2-\cos(1-x))^5+C}$

(p) $\displaystyle\int (1-\tfrac{1}{v})\cos(v-\ln(v))\,dv = \boxed{\sin(v-\ln(v))+C}$

(q) $\displaystyle\int \frac{t}{\sqrt{1-4t^2}}\,dt = \boxed{\tfrac{-1}{4}\sqrt{1-4t^2}+C}$

(r) $\displaystyle\int_0^{\pi/3} \big(3\sin(\tfrac{1}{2}x)+5\cos(x)\big)\,dx = \boxed{6-\frac{\sqrt{3}}{2}}$

(s) $\displaystyle\int \frac{e^{\tan(x)}}{\cos(x)^2}\,dx = \boxed{e^{\tan(x)} + C}$

(t) $\displaystyle\int_1^5 \frac{x^2+1}{x^3+3x}\,dx = \tfrac{1}{3}\ln(x^3+3x)\Big|_1^5 = \boxed{\frac{1}{3}\ln(35)}$

196. If $\displaystyle\int_9^{16} f(x)\,dx = 1$, calculate $\displaystyle\int_3^{10} f(x^2)\,x\,dx = \boxed{\frac{1}{2}}$

☆197. If $\displaystyle\int_0^1 f(x)\,dx = 7$ and $\displaystyle\int_0^2 f(x)\,dx = 11$, calculate $\displaystyle\int_0^1 f(2^x)\,2^x\,dx$. With $u = 2^x$ we have $u(0) = 1$ and $u(1) = 2$, so this is $\displaystyle\int_1^2 f(u)\tfrac{1}{\ln(2)}\,du = \boxed{\frac{4}{\ln(2)}}$

198. If $\dfrac{dv}{dx} = \sin(2x)$, what is one possibility for v? $\boxed{-\tfrac{1}{2}\cos(2x)}$

199. Fill in the missing parts of the table:

$f =$	$\sin(x)$	$\ln(x)$	x^3	$\boxed{-\frac{1}{2}\cos(x)}$	$\boxed{\frac{1}{2}x^2}$	$\boxed{\ln(x)}$
$df =$	$\cos(x)\,dx$	$\boxed{\frac{1}{x}\,dx}$	$\boxed{3x^2\,dx}$	$\sin(2x)\,dx$	$x\,dx$	$\frac{dx}{x}$

200. (a) Calculate the definite integral $\displaystyle\int_{\pi/4}^{3\pi/4} \frac{\cos x}{(\sin x)^3+1}\,dx$. $\boxed{0}$

☆(b) Find the indefinite integral $\displaystyle\int \frac{\cos x}{(\sin x)^3+1}\,dx$. This is a starred task. You are **not** expected to be able to do this indefinite integral for this class. But there is a formula if you are curious:

$$\boxed{\frac{1}{\sqrt{3}}\tan^{-1}\!\left(\tfrac{2}{\sqrt{3}}\sin(x) - \tfrac{1}{\sqrt{3}}\right) - \frac{1}{6}\ln\!\left(\sin^2(x) - \sin(x) + 1\right) + \frac{1}{3}\ln\!\left(\sin(x)+1\right)}$$

Integration by parts for indefinite integrals:

$$\int f'g\,dx \;=\; fg - \int fg'\,dx.$$

The formula "$\int u\,dv = uv - \int v\,du$" is in fact the same, with $u = g$ and $v = f$.

201. Use integration by parts to evaluate $\displaystyle\int 4xe^{2x}\,dx$. $\boxed{(2x-1)e^{2x} + C}$

202. Use integration by parts to find $\displaystyle\int \ln(x)\,dx$. Hint: $f' = 1$ and $g = \ln(x)$.

$f = x$, and $g' \, \mathrm{d}x = \frac{1}{x} \, \mathrm{d}x$, so

$$\begin{aligned}
\int \ln(x) 1 \, \mathrm{d}x + \int x \frac{1}{x} \, \mathrm{d}x &= x \ln(x) \\
\int \ln(x) \, \mathrm{d}x + \int 1 \, \mathrm{d}x &= x \ln(x) \\
\int \ln(x) \, \mathrm{d}x + x + C &= x \ln(x) \\
\int \ln(x) \, \mathrm{d}x &= \boxed{x \ln(x) - x + C}
\end{aligned}$$

203. Find the following indefinite integrals using integration by parts:

(a) $\int x \sin(x) \, \mathrm{d}x$

With $u = x$ and $\mathrm{d}v = \sin(x) \, \mathrm{d}x$, we have $\mathrm{d}u = \mathrm{d}x$ and $v = -\cos(x)$.

$$uv - \int v \, \mathrm{d}u = x(-\cos(x)) + \int \cos(x) \, \mathrm{d}x = \boxed{\sin(x) - x \cos(x) + C}.$$

(b) $\int x \cos(8x) \, \mathrm{d}x = \boxed{\frac{1}{8} \, x \sin(8x) + \frac{1}{64} \cos(8x) + C}$

(c) $\int \frac{\ln(x)}{x^5} \, \mathrm{d}x = \boxed{-\frac{\ln(x)}{4x^4} - \frac{1}{16x^4} + C}$

(d) $\int x^2 \cos(4x) \, \mathrm{d}x$

With $u = x^2$ and $\mathrm{d}v = \cos(4x) \, \mathrm{d}x$, we have $\mathrm{d}u = 2x \, \mathrm{d}x$ and $v = \frac{1}{4} \sin(4x)$.

$$\begin{aligned}
\int x^2 \cos(4x) \, \mathrm{d}x = uv - \int v \, \mathrm{d}u &= \tfrac{1}{4} x^2 \sin(4x) - \int \tfrac{1}{4} \sin(4x) 2x \, \mathrm{d}x \\
&= \tfrac{1}{4} x^2 \sin(4x) - \tfrac{1}{2} \int x \sin(4x) \, \mathrm{d}x
\end{aligned}$$

This new integral also requires integration by parts (it is extremely similar to parts a and b).

New $u = x$ and $\mathrm{d}v = \sin(4x) \, \mathrm{d}v$ gives $\mathrm{d}u = \mathrm{d}x$ and $v = \frac{-1}{4} \cos(4x)$, so

$$\begin{aligned}
\int x \sin(4x) \, \mathrm{d}x &= x(\tfrac{-1}{4} \cos(4x)) - \int \tfrac{-1}{4} \cos(4x) \, \mathrm{d}x \\
&= \tfrac{-1}{4} x \cos(4x) - \tfrac{1}{16} \sin(4x) + C
\end{aligned}$$

and then

$$\begin{aligned}
\int x^2 \cos(4x) \, \mathrm{d}x &= \tfrac{1}{4} x^2 \sin(4x) - \tfrac{1}{2} \left(\tfrac{-1}{4} x \cos(4x) - \tfrac{1}{16} \sin(4x) + C \right) \\
&= \tfrac{1}{4} x^2 \sin(4x) + \tfrac{1}{8} x \cos(4x) - \tfrac{1}{32} \sin(4x) + C \\
&= \boxed{\tfrac{1}{8} \, x \cos(4x) + (\tfrac{1}{4} x^2 - \tfrac{1}{32}) \sin(4x) + C}
\end{aligned}$$

(e) $\int (4x+12)e^{x/3}\,\mathrm{d}x$

$u = 4x+12$ and $\mathrm{d}v = e^{x/3}\,\mathrm{d}x$ gives $\mathrm{d}u = 4\,\mathrm{d}x$ and $v = 3e^{x/3}$.

$(4x+12)(3e^{x/3}) - \int 12e^{x/3}\,\mathrm{d}x = (12x+36)e^{x/3} - 36e^{x/3} + C = \boxed{12x\,e^{x/3} + C}$.

(f) $\int \cos(x)e^{2x}\,\mathrm{d}x$

$u = \cos(x)$ and $\mathrm{d}v = e^{2x}\,\mathrm{d}x$ gives $\mathrm{d}u = -\sin(x)\,\mathrm{d}x$ and $v = \frac{1}{2}e^{2x}$.

$$\int e^{2x}\cos(x)\,\mathrm{d}x = \tfrac{1}{2}\cos(x)e^{2x} - \int \tfrac{-1}{2}e^{2x}\sin(x)\,\mathrm{d}x$$
$$= \tfrac{1}{2}\cos(x)e^{2x} + \tfrac{1}{2}\int e^{2x}\sin(x)\,\mathrm{d}x.$$

We need parts again to do $\int e^{2x}\sin(x)\,\mathrm{d}x$.

New $u = \sin(x)$ and $\mathrm{d}v = e^{2x}$ (again) gives $\mathrm{d}u = \cos(x)\,\mathrm{d}x$ and $v = \frac{1}{2}e^{2x}$.

$$\int e^{2x}\sin(x)\,\mathrm{d}x = \tfrac{1}{2}\sin(x)e^{2x} - \int \tfrac{1}{2}e^{2x}\cos(x)\,\mathrm{d}x.$$

Therefore

$$\int e^{2x}\cos(x)\,\mathrm{d}x = \tfrac{1}{2}\cos(x)e^{2x} + \tfrac{1}{2}\left(\tfrac{1}{2}\sin(x)e^{2x} - \int \tfrac{1}{2}e^{2x}\cos(x)\,\mathrm{d}x\right)$$
$$\int \boldsymbol{e^{2x}\cos(x)\,\mathrm{d}x} = \tfrac{1}{2}\cos(x)e^{2x} + \tfrac{1}{4}\sin(x)e^{2x} - \tfrac{1}{4}\int \boldsymbol{e^{2x}\cos(x)\,\mathrm{d}x}$$
$$\tfrac{5}{4}\int \boldsymbol{e^{2x}\cos(x)\,\mathrm{d}x} = \tfrac{1}{2}\cos(x)e^{2x} + \tfrac{1}{4}\sin(x)e^{2x} + C$$
$$\int \boldsymbol{e^{2x}\cos(x)\,\mathrm{d}x} = \boxed{\tfrac{2}{5}\cos(x)e^{2x} + \tfrac{1}{5}\sin(x)e^{2x} + C}$$

204. Calculate the following definite integrals using integration by parts:

(a) $\int_0^6 (4x+12)e^{x/3}\,\mathrm{d}x$ We could use **Task 203(e)** and then

$$12x\,e^{x/3}\Big|_0^6 = 72\,e^2 - 0 = \boxed{72\,e^2}.$$

Alternatively, we can use the formula from the box, with $f = 4x+12$ and $g = 3e^{x/3}$:

$$\int_0^6 (4x+12)3e^{x/3}\,\mathrm{d}x + \int_0^6 3e^{x/3}4\,\mathrm{d}x = (4x+12)3e^{x/3}\Big|_{x=0}^{x=6}$$
$$\int_0^6 (4x+12)3e^{x/3}\,\mathrm{d}x + 36e^{x/3}\Big|_{x=0}^{x=6} = (4x+12)3e^{x/3}\Big|_{x=0}^{x=6}$$
$$\int_0^6 (4x+12)3e^{x/3}\,\mathrm{d}x + 36e^2 - 36 = 108e^2 - 36$$
$$\int_0^6 (4x+12)3e^{x/3}\,\mathrm{d}x = 108e^2 - 36e^2 = \boxed{72e^2}$$

(b) $\displaystyle\int_1^2 x\ln(x)\,\mathrm{d}x = \boxed{\ln(4) - \frac{3}{4}}$

(c) $\displaystyle\int_0^1 t\sin(\pi t)\,\mathrm{d}t = \boxed{\frac{1}{\pi}}$

(d) $\displaystyle\int_0^\pi x^4\cos(4x)\,\mathrm{d}x$

To find the antiderivative $F(x)$, use integration by parts four times (each time, x^4 will become a lower power until we have $\int k\cos(4x)\,\mathrm{d}x$).

$$\begin{aligned}
F(x) &= \int x^4\cos(4x)\,\mathrm{d}x \\
&= x^4\cdot\frac{1}{4}\sin(4x) - \int 4x^3\cdot\frac{1}{4}\sin(4x)\,\mathrm{d}x \\
&= \frac{1}{4}x^4\sin(4x) - \int x^3\sin(4x)\,\mathrm{d}x \\
&= \frac{1}{4}x^4\sin(4x) - \left(-\frac{1}{4}x^3\cos(4x) + \int\frac{3}{4}x^2\cos(4x)\,\mathrm{d}x\right) \\
&= \frac{1}{4}x^4\sin(4x) + \frac{1}{4}x^3\cos(4x) + \frac{3}{4}\int x^2\cos(4x)\,\mathrm{d}x \\
&= \frac{1}{4}x^4\sin(4x) + \frac{1}{4}x^3\cos(4x) + \left(-\frac{3}{16}x^2\sin(4x) - \int -\frac{3}{8}x\sin(4x)\,\mathrm{d}x\right) \\
&= \frac{1}{4}x^4\sin(4x) + \frac{1}{4}x^3\cos(4x) - \frac{3}{16}x^2\sin(4x) + \int\frac{3}{8}x\sin(4x)\,\mathrm{d}x \\
&= \frac{1}{4}x^4\sin(4x) + \frac{1}{4}x^3\cos(4x) - \frac{3}{16}x^2\sin(4x) + \left(-\frac{3}{32}x\cos(4x) - \int -\frac{3}{32}\cos(4x)\,\mathrm{d}x\right) \\
&= \frac{1}{4}x^4\sin(4x) + \frac{1}{4}x^3\cos(4x) - \frac{3}{16}x^2\sin(4x) - \frac{3}{32}x\cos(4x) + \int\frac{3}{32}\cos(4x)\,\mathrm{d}x \\
F(x) &= \frac{1}{4}x^4\sin(4x) + \frac{1}{4}x^3\cos(4x) - \frac{3}{16}x^2\sin(4x) - \frac{3}{32}x\cos(4x) + \frac{3}{128}\sin(4x) + C
\end{aligned}$$

Then the definite integral is calculated using

$$\begin{aligned}
F(\pi) &= \frac{1}{4}\pi^4\sin(4\pi) + \frac{1}{4}\pi^3\cos(4\pi) - \frac{3}{16}\pi^2\sin(4\pi) - \frac{3}{32}\pi\cos(4\pi) + \frac{3}{128}\sin(4\pi) \\
&= 0 + \frac{1}{4}\pi^3 - 0 - \frac{3}{32}\pi + 0 = \frac{\pi^3}{4} - \frac{3\pi}{32} \\
F(0) &= \frac{1}{4}0^4\sin(0) + \frac{1}{4}0^3\cos(0) - \frac{3}{16}0^2\sin(0) - \frac{3}{32}0\cos(0) + \frac{3}{128}\sin(0) = 0
\end{aligned}$$

so $F(\pi) - F(0) = F(\pi) = \boxed{\frac{\pi^3}{4} - \frac{3\pi}{32}}$ or $\dfrac{\pi(8\pi^2-3)}{32}$.

Previously the file incorrectly said that $\frac{\pi}{8}$ was the answer.

☆205. Prove that $\displaystyle\int_1^\pi \ln(x)\cos(x)\,\mathrm{d}x = \int_1^\pi \frac{-\sin(x)}{x}\,\mathrm{d}x$.

$$\int_1^\pi \ln(x)\cos(x)\,\mathrm{d}x + \int_1^\pi \frac{\sin(x)}{x}\,\mathrm{d}x = \ln(x)\sin(x)\Big|_{x=0}^{x=\pi} = 0$$

206. Find $\int 4x\cos(2-3x)\,dx$. With $u = 4x$ and $dv = \cos(2-3x)\,dx$, we have $du = 4\,dx$ and $v = \frac{-1}{3}\sin(2-3x)$.

$$uv - \int v\,du = 4x(\tfrac{-1}{3}\sin(2-3x)) - \int 4(\tfrac{-1}{3}\sin(2-3x))\,dx$$
$$= \boxed{\tfrac{-4}{3}\sin(2-3x) + \tfrac{4}{9}\cos(2-3x) + C}.$$

207. Try each of the following methods to find $\int \sin(x)\cos(x)\,dx$.
(They are all possible.)

(a) Substitue $u = \sin(x)$, so $du = \cos(x)\,dx$ and the integral is $\int u\,du$.

(b) Substitue $u = -\cos(x)$, so $du = \sin(x)\,dx$, and the integral is $\int -u\,du$.

(c) Substitute $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$, so the integral is $\frac{1}{2}\int \sin(2x)\,dx$.

(d) Do integration by parts with $g = \sin(x)$ and $f' = \cos(x)$.

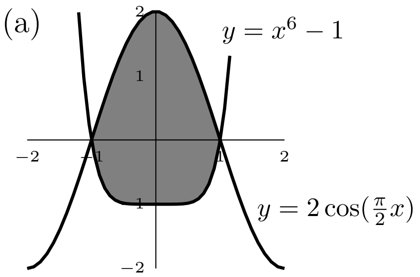
(e) Do integration by parts with $g = \cos(x)$ and $f' = \sin(x)$.

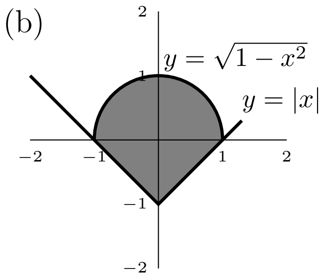
(f) Compare your answers to parts (a) through (e).
See `https://youtu.be/-JR9-dgU7tU?t=520`

208. Find $\int (2-3x)\cos(4x)\,dx = \boxed{\dfrac{1}{2}\sin(4x) - \dfrac{3}{4}x\sin(4x) - \dfrac{3}{16}\cos(4x) + C}$

The area between two curves of the form $y = f(x)$ is $\displaystyle\int_{\text{left}}^{\text{right}} \big(\text{top}(x) - \text{bottom}(x)\big)\,dx$.

209. Give the areas of the two shapes below:





(a) $\displaystyle\int_{-1}^{1} \big(2\cos(\tfrac{\pi}{2}x) - (x^6-1)\big)\,dx = \Big[\tfrac{4}{\pi}\sin(\tfrac{\pi}{2}x) - \tfrac{1}{6}x^7 + x\Big]_{x=-1}^{x=1} = \boxed{\dfrac{8}{\pi} + \dfrac{12}{7}} \approx 4.26.$

(b) The area of a semicircle is $\frac{1}{2}\pi r^2 = \frac{\pi}{2} \approx 1.5708$. Area of triangle is $\frac{1}{2}hb = \frac{1}{2}(2)(1) = 1$. Total area is $\boxed{1 + \dfrac{\pi}{2} \approx 2.5708}$. This is also $\displaystyle\int_{-1}^{1}\big(\sqrt{1-x^2} - |x|\big)\,dx$, but it is difficult to get a formula for the anti-derivative.

210. Find the area of the region bounded by $y = e^x$, $y - x = 5$, $x = -4$, and $x = 0$?

(That is, find the area between $y = e^x$ and $y = x + 5$ with $-4 \le x \le 0$).

$\displaystyle\int_{-4}^{0}\big((x+5) - e^x\big)\,dx = \boxed{11 + e^{-4}}$

211. What is the area of the region bounded by the curves $y + x^4 = 20$ and $y = 4$?

$$\int_{-2}^{2} \big((20 - x^4) - 4\big)\, \mathrm{d}x = \boxed{\frac{256}{5}}$$

212. Calculate each of the following integrals.
Some* require substitution, some** require parts, and some do not need either.

(a) $\displaystyle\int (x^4 + x^{1/2} + 4 + x^{-1})\, \mathrm{d}x = \boxed{\tfrac{1}{5}x^5 + \tfrac{1}{3}x^{3/2} + \ln|x| + C}$

(b) $\displaystyle\int \left((x^2)^2 + \sqrt{x} + \frac{\ln(81)}{\ln(3)} + \frac{1}{x}\right) \mathrm{d}x = \boxed{\tfrac{1}{5}x^5 + \tfrac{1}{3}x^{3/2} + \ln|x| + C}$

(c) $\displaystyle\int (t + e^t)\, \mathrm{d}t = \boxed{\frac{t^2}{2} + e^t + C}$

(d) $\displaystyle\int (t \cdot e^t)\, \mathrm{d}t = \boxed{(t-1)e^t + C}$

(e) $\displaystyle\int (t^3 + e^{3t})\, \mathrm{d}t = \boxed{\frac{t^4}{4} + \frac{e^{3t}}{3} + C}$

(f) $\displaystyle\int (t^3 \cdot e^{3t})\, \mathrm{d}t = \boxed{\tfrac{1}{27}e^{3t}(9t^3 - 9t^2 + 6t - 2) + C}$

(g) $\displaystyle\int \frac{x}{x^2+1}\, \mathrm{d}x = \boxed{\tfrac{1}{2}\ln(x^2+1) + C}$

(h) $\displaystyle\int \frac{x}{x^2-1}\, \mathrm{d}x = \boxed{\tfrac{1}{2}\ln|x^2-1| + C}$

(i) $\displaystyle\int \frac{x^2-1}{x}\, \mathrm{d}x = \boxed{\tfrac{1}{2}x^2 - \ln|x| + C}$

(j) $\displaystyle\int \frac{1}{x^2-1}\, \mathrm{d}x = \boxed{\tfrac{1}{2}\ln|1-x| - \tfrac{1}{2}\ln|x+1| + C}$

(k) $\displaystyle\int \frac{1}{x^2+1}\, \mathrm{d}x = \boxed{\operatorname{arctg}(x) + C}$ (See List 8)

(ℓ) $\displaystyle\int \frac{y}{\sqrt{y^2+1}}\, \mathrm{d}y = \boxed{\sqrt{y^2+1} + C}$

☆ (m) $\displaystyle\int \frac{1}{\sqrt{y^2+1}}\, \mathrm{d}y = \boxed{\ln(y + \sqrt{y^2+1}) + C}$

(n) $\displaystyle\int t\ln(t)\, \mathrm{d}t = \boxed{\tfrac{1}{2}t^2\ln(t) - \tfrac{1}{4}t^4 + C}$

(o) $\displaystyle\int \frac{3t-12}{\sqrt{t^2-8t+6}}\, \mathrm{d}t = \boxed{3\sqrt{t^2-8t+6} + C}$

(p) $\displaystyle\int \frac{1}{\sqrt{x-1}}\, \mathrm{d}x = \boxed{2\sqrt{x-1} + C}$

(q) $\displaystyle\int \frac{x}{\sqrt{x-1}}\, \mathrm{d}x = \boxed{\tfrac{2}{3}(x+2)\sqrt{x-1} + C}$

(r) $\displaystyle\int y^3\, \mathrm{d}y = \boxed{\tfrac{1}{4}y^4 + C}$

(s) $\displaystyle\int y(y+1)(y-1)\,\mathrm{d}y = \boxed{\tfrac{1}{4}y^4 - \tfrac{1}{2}y^2 + C}$

(t) $\displaystyle\int x\sin(2x)\,\mathrm{d}x = \boxed{\frac{1}{4}\sin(2x) - \frac{1}{2}x\cos(2x) + C}$

(u) $\displaystyle\int x^3\sin(2x^4)\,\mathrm{d}x = \boxed{\tfrac{-1}{8}\cos(2x^4) + C}$

☆ (v) $\displaystyle\int x^7\sin(2x^4)\,\mathrm{d}x$

This actually requires both parts and substitution!

If we substitute $w = 2x^4$ then $\mathrm{d}w = 8x^3\,\mathrm{d}x$ and, using $x^7 = x^4 \cdot x^3$, we have

$$\int x^7\sin(2x^4)\,\mathrm{d}x = \int x^4 \cdot \sin(2x^4) \cdot x^3\,\mathrm{d}x$$
$$= \int (\tfrac{1}{2}w) \cdot \sin(w) \cdot \tfrac{1}{8}\,\mathrm{d}w = \frac{1}{16}\int w\sin(w)\,\mathrm{d}w.$$

Using integration by parts on $\int w\sin(w)\,\mathrm{d}w$ is exactly like **Task 203(a)**, so we get $\int w\sin(w)\,\mathrm{d}w = \sin(w) - w\cos(w) + C$ and

$$\tfrac{1}{16}\sin(w) - \tfrac{1}{16}w\cos(w) + C = \boxed{\tfrac{1}{16}\sin(2x^4) - \tfrac{1}{8}x^4\cos(2x^4) + C}.$$

(w) $\displaystyle\int \frac{3x}{1+x^4}\,\mathrm{d}x = \boxed{\tfrac{3}{2}\operatorname{arctg}(x^2) + C}$ (See List 8)

(x) $\displaystyle\int e^{5x}\cos(e^{5x})\,\mathrm{d}x = \boxed{\tfrac{1}{5}\sin(e^{5x}) + C}$

☆ (y) $\displaystyle\int x^5\cos(x)\,\mathrm{d}x = \boxed{(x^5 - 20x^3 + 120x)\sin(x) + (5x^4 - 60x^2 + 120)\cos(x) + C}$

(z) $\displaystyle\int e^{8\ln(t)}\,\mathrm{d}t = \boxed{\tfrac{1}{9}t^9 + C}$

* g, h, m, o, p, q, u, w, x. ** d, f, ℓ, n, t, v, y.