Math for Management, Winter 2023

List 7

Integrals (definite, indefinite, u-sub., parts)

An **indefinite integral** describes all the anti-derivatives of a function. We write

$$\int f(x)\,dx = F(x) + C,$$

where $F(x)$ is any function for which $F'(x) = f(x)$.

175. Find $\displaystyle\int (6x^5 + 7x^3 - 9)\,\mathrm{d}x$.

176. Find $\displaystyle\int (6u^5 + 7u^3 - 9)\,\mathrm{d}u$.

177. Give each of the following indefinite integrals using basic derivative knowledge:

(a) $\displaystyle\int x^{372.5}\,\mathrm{d}x$ (c) $\displaystyle\int e^x\,\mathrm{d}x$ (e) $\displaystyle\int -\sin(x)\,\mathrm{d}x$ (g) $\displaystyle\int \cos(x)\,\mathrm{d}x$

(b) $\displaystyle\int \frac{1}{x}\,\mathrm{d}x$ (d) $\displaystyle\int 97^x\,\mathrm{d}x$ (f) $\displaystyle\int \sin(x)\,\mathrm{d}x$ (h) $\displaystyle\int 5t^9\,\mathrm{d}t$

178. If $u = 6x^2 - 5$, give a formula for $\mathrm{d}u$ (this formula will have x and $\mathrm{d}x$ in it) and a formula for $\mathrm{d}x$ (this formula will have x and $\mathrm{d}u$ in it).

Substitution: $\displaystyle\int f(u(x)) \cdot u'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$

179. (a) Re-write $\displaystyle\int \frac{x}{(6x^2-5)^3}\,\mathrm{d}x$ as $\displaystyle\int ...\,\mathrm{d}u$ using the substitution $u = 6x^2 - 5$.

(b) Find $\displaystyle\int \frac{x}{(6x^2-5)^3}\,\mathrm{d}x$. (Your final answer should not have u at all.)

180. (a) Re-write $\displaystyle\int x^3\sin(x^4)\,\mathrm{d}x$ as $\displaystyle\int ...\,\mathrm{d}u$ using the substitution $u = x^4$.

(b) Find $\displaystyle\int x^3\sin(x^4)\,\mathrm{d}x$.

181. (a) Re-write $\displaystyle\int x\sin(x^4)\,\mathrm{d}x$ as $\displaystyle\int ...\,\mathrm{d}u$ using the substitution $u = x^2$.

☆(b) Find $\displaystyle\int x\sin(x^4)\,\mathrm{d}x$.

182. Give $\displaystyle\int \frac{2x^3 - 7x + 3}{5x^4 - 35x^2 + 30x + 125}\,\mathrm{d}x$ using substitution.

183. Find $\displaystyle\int \cot(x)\,\mathrm{d}x$ using substitution. Hint: $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$.

The notation $\boldsymbol{g(x)}\Big|_{\boldsymbol{x=a}}^{\boldsymbol{x=b}}$ or $g\Big|_a^b$ or $\Big[g(x)\Big]_a^b$ means to do the subtraction $g(b) - g(a)$.

184. Calculate $\frac{1}{3}x^3\Big|_{x=1}^{x=2}$.

185. Calculate $(x^3 + \frac{1}{2}x)\Big|_{x=1}^{x=5} \;=\; \Big[x^3 + \frac{1}{2}x\Big]_{x=1}^{x=5}$.

186. Calculate $\dfrac{1-x}{e^x}\Bigg|_{x=0}^{x=1}$.

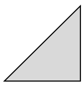
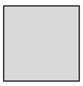
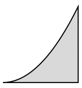
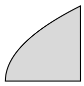
The **definite integral** $\displaystyle\int_a^b f(x)\,dx$, spoken as "the integral from a to b of $f(x)$ with respect to x", is the (signed) area of the region with $x = a$ on the left, $x = b$ on the right, $y = 0$ at the bottom, and $y = f(x)$ at the top (but if $f(x) < 0$ for some x or if $b < a$ then it's possible for the area to be negative).

The **Newton–Leibniz Theorem** (**NL**, also called the "Fundamental Theorem of Calculus" or FTC) says that

$$\int_a^b f(x)\,\mathrm{d}x \;=\; F(x)\Big|_{x=a}^{x=b} \;=\; F(b) - F(a),$$

where $F(x)$ is any function for which $F'(x) = f(x)$.

187. Match the shapes (a)-(d) with the integral (I)-(IV) that is most likely to calculate its area.

(a)  (b)  (c)  (d) 

(I) $\displaystyle\int_0^1 \sqrt{x}\,\mathrm{d}x$ (II) $\displaystyle\int_0^1 x\,\mathrm{d}x$ (III) $\displaystyle\int_0^1 x^2\,\mathrm{d}x$ (IV) $\displaystyle\int_0^1 1\,\mathrm{d}x$

188. Calculate $\displaystyle\int_1^2 x^2\,\mathrm{d}x$ using the Newton–Leibniz Theorem.

189. Write, in symbols, the integral from zero to six of x^2 with respect to x, then find the value of that definite integral.

190. Evaluate (meaning find of the value of) the following definite integrals using common area formulas.

(a) $\displaystyle\int_3^9 2\,\mathrm{d}x$ (c) $\displaystyle\int_0^5 x\,\mathrm{d}x$ (e) $\displaystyle\int_0^5 3x\,\mathrm{d}x$ (g) $\displaystyle\int_{-4}^4 \sqrt{16-x^2}\,\mathrm{d}x$

(b) $\displaystyle\int_3^9 -2\,\mathrm{d}x$ (d) $\displaystyle\int_{-2}^4 |x|\,\mathrm{d}x$ (f) $\displaystyle\int_1^5 3x\,\mathrm{d}x$ (h) $\displaystyle\int_0^7 \sqrt{49-x^2}\,\mathrm{d}x$

191. Evaluate the following definite integrals using the FTC. Your answer for each should be a number.

(a) $\displaystyle\int_{-3}^9 2\,\mathrm{d}x$ (c) $\displaystyle\int_1^{12} \frac{1}{x}\,\mathrm{d}x$ (e) $\displaystyle\int_0^\pi \sin(t)\,\mathrm{d}t$

(b) $\displaystyle\int_1^5 3x\,\mathrm{d}x$ (d) $\displaystyle\int_0^9 (x^3 - 9x)\,\mathrm{d}x$ (f) $\displaystyle\int_2^8 3\sqrt{u}\,\mathrm{d}u$

(g) $\int_0^1 (e^x + x^e)\,\mathrm{d}x$

(h) $\int_{-1}^1 x^2\,\mathrm{d}x$

(i) $\int_1^3 t\,\mathrm{d}t$

(j) $\int_9^9 \sin(x^2)\,\mathrm{d}x$

(k) $\int_0^5 \cos(x)\,\mathrm{d}x$

192. If $\int_1^4 f(x)\,\mathrm{d}x = 12$ and $\int_1^6 f(x)\,\mathrm{d}x = 15$, what is the value of $I = \int_4^6 f(x)\,\mathrm{d}x$?

193. If $\int_0^1 f(x)\,\mathrm{d}x = 7$ and $\int_0^1 g(x)\,\mathrm{d}x = 3$, calculate each of the following or say that there is not enough information to possibly do the calculation.

(a) $\int_0^1 \big(f(x) + g(x)\big)\,\mathrm{d}x$

(b) $\int_0^1 \big(f(x) - g(x)\big)\,\mathrm{d}x$

(c) $\int_0^1 \big(f(x) \cdot g(x)\big)\,\mathrm{d}x$

(d) $\int_0^1 \big(5f(x)\big)\,\mathrm{d}x$

(e) $\int_0^1 \big(f(x)^5\big)\,\mathrm{d}x$

194. Which of the following has the same value as $\int_2^4 \frac{3x^2 - 2}{\ln(x^3 - 2x + 1)}\,\mathrm{d}x$?

(A) $\int_5^{57} \frac{1}{\ln(u)}\,\mathrm{d}u$ (B) $\int_2^4 \frac{1}{\ln(u)}\,\mathrm{d}u$ (C) $\int_{10}^{46} \frac{1}{\ln(u)}\,\mathrm{d}u$ (D) $\int_1^2 \frac{1}{\ln(u)}\,\mathrm{d}u$

195. Find the following integrals using substitution:

(a) $\int (5 - x)^{10}\,\mathrm{d}x$

(b) $\int_1^3 \frac{x}{(6x^2 - 5)^3}\,\mathrm{d}x$

(c) $\int \sqrt{4x + 3}\,\mathrm{d}x$

(d) $\int_0^{\sqrt{\pi}} x \sin(x^2)\,\mathrm{d}x$

(e) $\int \frac{5}{4x + 9}\,\mathrm{d}x$

(f) $\int \frac{5x}{4x^2 + 9}\,\mathrm{d}x$

☆ (g) $\int \frac{5}{4x^2 + 9}\,\mathrm{d}x$

(h) $\int \frac{\sin(\ln(x))}{x}\,\mathrm{d}x$

☆ (i) $\int_0^9 \sqrt{4 - \sqrt{x}}\,\mathrm{d}x$

(j) $\int x^3 \cos(2x^4)\,\mathrm{d}x$

(k) $\int e^{t^5} t^4\,\mathrm{d}t$

(l) $\int \frac{(\ln(x))^2}{5x}\,\mathrm{d}x$

(m) $\int \frac{1}{x \ln(x)}\,\mathrm{d}x$

(n) $\int_0^{\pi/2} \sin(x) \cos(x)\,\mathrm{d}x$

(o) $\int \sin(1 - x)(2 - \cos(1 - x))^4\,\mathrm{d}x$

(p) $\int (1 - \frac{1}{v}) \cos(v - \ln(v))\,\mathrm{d}v$

(q) $\int \frac{t}{\sqrt{1 - 4t^2}}\,\mathrm{d}t$

(r) $\int_0^{\pi/3} \big(3 \sin(\tfrac{1}{2}x) + 5 \cos(x)\big)\,\mathrm{d}x$

(s) $\int \frac{e^{\tan(x)}}{\cos(x)^2}\,\mathrm{d}x$

(t) $\int_1^5 \frac{x^2 + 1}{x^3 + 3x}\,\mathrm{d}x$

196. If $\int_9^{16} f(x)\,\mathrm{d}x = 1$, calculate $\int_3^{10} f(x^2)\,x\,\mathrm{d}x$.

☆197. If $\int_0^1 f(x)\,\mathrm{d}x = 7$ and $\int_0^2 f(x)\,\mathrm{d}x = 11$, calculate $\int_0^1 f(2^x)\,2^x\,\mathrm{d}x$.

198. If $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin(2x)$, what is one possibility for v?

199. Fill in the missing parts of the table:

$f =$	$\sin(x)$	$\ln(x)$	x^3			
$\mathrm{d}f =$	$\cos(x)\,\mathrm{d}x$			$\sin(2x)\,\mathrm{d}x$	$x\,\mathrm{d}x$	$\frac{\mathrm{d}x}{x}$

200. (a) Calculate the definite integral $\int_{\pi/4}^{3\pi/4} \frac{\cos x}{(\sin x)^3 + 1}\,\mathrm{d}x$.

☆(b) Find the indefinite integral $\int \frac{\cos x}{(\sin x)^3 + 1}\,\mathrm{d}x$.

Integration by parts for indefinite integrals:

$$\int fg'\,\mathrm{d}x + \int gf'\,\mathrm{d}x = fg.$$

201. Use integration by parts to evaluate $\int 4xe^{2x}\,\mathrm{d}x$.

202. Use integration by parts to find $\int \ln(x)\,\mathrm{d}x$. Hint: $f = \ln(x)$ and $g' = 1$.

203. Find the following indefinite integrals using integration by parts:

(a) $\int x\sin(x)\,\mathrm{d}x$

(b) $\int x\cos(8x)\,\mathrm{d}x$

(c) $\int \frac{\ln(x)}{x^5}\,\mathrm{d}x$

(d) $\int x^2\cos(4x)\,\mathrm{d}x$

(e) $\int (4x+12)e^{x/3}\,\mathrm{d}x$

(f) $\int \cos(x)e^{2x}\,\mathrm{d}x$

For definite integrals, $\int_a^b fg'\,\mathrm{d}x + \int_a^b f'g\,\mathrm{d}x = fg\Big|_{x=a}^{x=b}$.

204. Calculate the following definite integrals using integration by parts:

(a) $\int_0^6 (4x+12)e^{x/3}\,\mathrm{d}x$

(b) $\int_1^2 x\ln(x)\,\mathrm{d}x$

(c) $\int_0^1 t\sin(\pi t)\,\mathrm{d}t$

(d) $\int_0^\pi x^4\cos(4x)\,\mathrm{d}x$

☆205. Prove that $\int_1^\pi \ln(x)\cos(x)\,\mathrm{d}x = \int_1^\pi \frac{-\sin(x)}{x}\,\mathrm{d}x$.

206. Find $\displaystyle\int 4x\cos(2-3x)\,\mathrm{d}x$.

207. Try each of the following methods to find $\displaystyle\int \sin(x)\cos(x)\,\mathrm{d}x$.
(They are all possible.)

(a) Substitue $u=\sin(x)$, so $\mathrm{d}u=\cos(x)\,\mathrm{d}x$ and the integral is $\displaystyle\int u\,\mathrm{d}u$.

(b) Substitue $u=-\cos(x)$, so $\mathrm{d}u=\sin(x)\,\mathrm{d}x$, and the integral is $\displaystyle\int -u\,\mathrm{d}u$.

(c) Substitute $\sin(x)\cos(x)=\frac{1}{2}\sin(2x)$, so the integral is $\displaystyle\frac{1}{2}\int \sin(2x)\,\mathrm{d}x$.

(d) Do integration by parts with $u=\sin(x)$ and $\mathrm{d}v=\cos(x)\,\mathrm{d}x$.

(e) Do integration by parts with $u=\cos(x)$ and $\mathrm{d}v=\sin(x)\,\mathrm{d}x$.

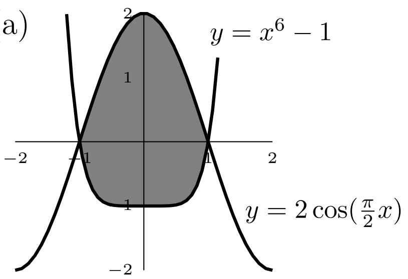
(f) Compare your answers to parts (a) through (e).

208. Find $\displaystyle\int (2-3x)\cos(4x)\,\mathrm{d}x$.

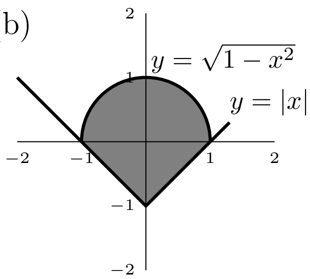
The area between two curves of the form $y=f(x)$ is $\displaystyle\int_{\text{left}}^{\text{right}} \big(\text{top}(x)-\text{bottom}(x)\big)\,\mathrm{d}x$.

209. Give the areas of the two shapes below:

(a)



(b)



210. Find the area of the region bounded by $y=e^x$, $y-x=5$, $x=-4$, and $x=0$?

(That is, find the area between $y=e^x$ and $y=x+5$ with $-4\le x\le 0$).

211. What is the area of the region bounded by the curves $y+x^4=20$ and $y=4$?

212. Calculate each of the following integrals.
Some* require substitution, some** require parts, and some do not need either.

(a) $\displaystyle\int (x^4 + x^{1/2} + 4 + x^{-1})\,\mathrm{d}x$

(b) $\displaystyle\int \left((x^2)^2 + \sqrt{x} + \frac{\ln(81)}{\ln(3)} + \frac{1}{x}\right)\mathrm{d}x$

(c) $\displaystyle\int (t + e^t)\,\mathrm{d}t$

(d) $\displaystyle\int (t \cdot e^t)\,\mathrm{d}t$

(e) $\displaystyle\int (t^3 + e^{3t})\,\mathrm{d}t$

(f) $\displaystyle\int (t^3 \cdot e^{3t})\,\mathrm{d}t$

(g) $\displaystyle\int \frac{x}{x^2+1}\,\mathrm{d}x$

(h) $\displaystyle\int \frac{x}{x^2-1}\,\mathrm{d}x$

(i) $\displaystyle\int \frac{x^2-1}{x}\,\mathrm{d}x$

(j) $\displaystyle\int \frac{1}{x^2-1}\,\mathrm{d}x$

(k) $\displaystyle\int \frac{1}{x^2+1}\,\mathrm{d}x$

(ℓ) $\displaystyle\int \frac{y}{\sqrt{y^2+1}}\,\mathrm{d}y$

☆ (m) $\displaystyle\int \frac{1}{\sqrt{y^2+1}}\,\mathrm{d}y$

(n) $\displaystyle\int t\ln(t)\,\mathrm{d}t$

(o) $\displaystyle\int \frac{3t-12}{\sqrt{t^2-8t+6}}\,\mathrm{d}t$

(p) $\displaystyle\int \frac{1}{\sqrt{x-1}}\,\mathrm{d}x$

(q) $\displaystyle\int \frac{x}{\sqrt{x-1}}\,\mathrm{d}x$

(r) $\displaystyle\int y^3\,\mathrm{d}y$

(s) $\displaystyle\int y(y+1)(y-1)\,\mathrm{d}y$

(t) $\displaystyle\int x\sin(2x)\,\mathrm{d}x$

(u) $\displaystyle\int x^3\sin(2x^4)\,\mathrm{d}x$

☆ (v) $\displaystyle\int x^7\sin(2x^4)\,\mathrm{d}x$

(w) $\displaystyle\int \frac{3x}{1+x^4}\,\mathrm{d}x$

(x) $\displaystyle\int e^{5x}\cos(e^{5x})\,\mathrm{d}x$

☆ (y) $\displaystyle\int x^5\cos(x)\,\mathrm{d}x$

(z) $\displaystyle\int e^{8\ln(t)}\,\mathrm{d}t$

* g, h, m, o, p, q, u, w, x. ** d, f, ℓ, n, t, v, y.