Math for Management, Winter 2023

List 8

More integrals, functions of two variables

The "arctangent" (or "inverse tangent") function has the following properties:

$$\operatorname{arctg}(0) = 0, \qquad \operatorname{arctg}(1) = \frac{\pi}{4}, \qquad \big(\operatorname{arctg}(x)\big)' = \frac{1}{x^2+1}.$$

213. Give the following integrals (arctg will appear somewhere in each answer):

(a) $\displaystyle\int \frac{5}{x^2+1}\,dx = \boxed{5\operatorname{arctg}(x) + C}$

(b) $\displaystyle\int \frac{5}{x^2+2}\,dx = \boxed{\tfrac{5}{\sqrt{2}}\operatorname{arctg}(\tfrac{x}{\sqrt{2}}) + C}$

(c) $\displaystyle\int \frac{x^2+2}{x^2+3}\,dx = \boxed{x - \tfrac{1}{\sqrt{3}}\operatorname{arctg}(\tfrac{x}{\sqrt{3}}) + C}$

(d) $\displaystyle\int \frac{x}{x^4+1}\,dx = \boxed{\tfrac{1}{2}\operatorname{arctg}(x^2) + C}$

214. Integrate by parts:

(a) $\displaystyle\int x^2 e^x\,dx = \boxed{e^x(x^2-2x+2) + C}$

(b) $\displaystyle\int x\ln x\,dx = \boxed{\frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C}$

(c) $\displaystyle\int \sqrt{x}\ln x\,dx = \boxed{\frac{2}{9}x^{3/2}(3\ln(x) - 2) + C}$

(d) $\displaystyle\int \frac{\ln x}{x^2}\,dx = \boxed{\frac{-1-\ln x}{x} + C}$

(e) $\displaystyle\int (\ln x)^2\,dx = \boxed{2x + x\ln^2(x) - 2x\ln x + C}$

(f) $\displaystyle\int \ln x\,dx = \boxed{x\ln(x) - x + C}$

(g) $\displaystyle\int \operatorname{arctg} x\,dx = \boxed{x\operatorname{arctg}(x) - \frac{1}{2}\ln(x^2+1) + C}$

(h) $\displaystyle\int_0^1 x^2\operatorname{arctg} x\,dx = \boxed{\frac{1}{3}x^3\operatorname{arctg}(x) - \frac{x^2}{6} + \frac{1}{6}\ln\left(x^2+1\right)}$

(i) $\displaystyle\int_1^e \left(\frac{\ln x}{x}\right)^2 dx = \boxed{\frac{-(\ln x)^2 - 2\ln(x) - 2}{x}}$

215. Integrate using substitution:

(a) $\displaystyle\int x\sqrt{x^2+1}\,dx = \boxed{\frac{1}{3}(x^2+1)^{3/2} + C}$

(b) $\displaystyle\int (5-3x)^{10}\,dx = \boxed{-(1/33)(5-3x)^11 + C}$ Earlier version of the file had $\int (5-3x)^{10} x\,dx$, which is a huge polynomial $\frac{19683}{4}x^{12} + \cdots\cdots + \frac{9765625}{2}x^2 + C$.

(c) $\displaystyle\int \sqrt{a+bx}\,dx = \boxed{\frac{2}{3b}(a+bx)^{3/2} + C}$

(d) $\displaystyle\int xe^{x^2}\,\mathrm{d}x = \boxed{\frac{e^{x^2}}{2}+C}$

(e) $\displaystyle\int \frac{\ln^2 x}{x}\,\mathrm{d}x = \boxed{\frac{\ln^3(x)}{3}+C}$

(f) $\displaystyle\int \frac{\ln x}{x}\,\mathrm{d}x = \boxed{\frac{\ln^2(x)}{2}+C}$

(g) $\displaystyle\int_0^4 \frac{\mathrm{d}x}{1+\sqrt{x}} = \left[2\sqrt{x}-2\ln(\sqrt{x}+1)\right]_{x=0}^{x=4} = \boxed{4-\ln(9)}$

(h) $\displaystyle\int_0^2 x^2\cdot 2^{x^3} = \left.\frac{2^{x^3}}{\ln 8}\right|_{x=0}^{x=2} = \boxed{\frac{255}{\ln 8}}$

216. Calculate the following indefinite integrals. You will have to decide what method (e.g., algebra simplification, parts, substitution) to use.

(a) $\displaystyle\int (3x^3+2\sqrt{x}-1)\,\mathrm{d}x = \boxed{\frac{4x^{3/2}}{3}+\frac{3x^4}{4}-x+C}$

(b) $\displaystyle\int x(x-1)(x-2)\,\mathrm{d}x = \boxed{\frac{x^4}{4}-x^3+x^2+C}$

(c) $\displaystyle\int \frac{3\sqrt[3]{x}-3}{x}\,\mathrm{d}x = \boxed{9\sqrt[3]{x}-3\ln x+C}$

(d) $\displaystyle\int \frac{x^2+2}{x^2+1}\,\mathrm{d}x = \boxed{x+\operatorname{arctg}(x)+C}$

(e) $\displaystyle\int \frac{x^3+8}{x^2}\,\mathrm{d}x = \boxed{\frac{x^2}{2}-\frac{8}{x}+C}$

(f) $\displaystyle\int \frac{x^2}{x^3+8}\,\mathrm{d}x = \boxed{\frac{1}{3}\ln(x^3+8)+C}$

(g) $\displaystyle\int (9x^2-x+1)^2\,\mathrm{d}x = \boxed{\frac{81}{5}x^5-\frac{9}{2}x^4+\frac{19}{3}x^3-x^2+x+C}$

(h) $\displaystyle\int \frac{e^x-2^x}{5^x}\,\mathrm{d}x = \boxed{\frac{\left(\frac{2}{5}\right)^x}{\ln\left(\frac{5}{2}\right)}+\frac{\left(\frac{e}{5}\right)^x}{\ln\left(\frac{e}{5}\right)}+C}$

217. Compute the following definite integrals:

(a) $\displaystyle\int_0^2 \frac{3x-1}{3x+1}\,\mathrm{d}x = \boxed{2-\frac{2}{3}\ln(7)}$

(b) $\displaystyle\int_2^3 \frac{\mathrm{d}x}{x^2+2x+1} = \boxed{\frac{1}{12}}$ Earlier version of file had $\int_{-3}^{2}\frac{\mathrm{d}x}{x^2+2x+1}$.

(c) $\displaystyle\int_0^2 \frac{x}{e^x}\,\mathrm{d}x = \boxed{1-\frac{3}{e^2}}$

(d) $\displaystyle\int_{-1}^2 |x|\,\mathrm{d}x\ \int_{-1}^0 |x|\,\mathrm{d}x+\int_0^2 |x|\,\mathrm{d}x = \int_{-1}^0 (-x)\,\mathrm{d}x+\int_0^2 x\,\mathrm{d}x = \frac{1}{2}+2 = \boxed{\frac{5}{2}}.$

218. Examine the graphs of sections of the function $z = z(x, y)$ and based on that draw the graphs of the function:

(a) $3x + 2y + z - 6 = 0$ This is a plane.

(b) $z^2 = x^2 + y^2$ This is a cone.

(c) $z = x^2 + y^2$ This is a "paraboloid".

> The point $(x, y) = (a, b)$ is a **stationary point** of $f(x, y)$ if both $f'_x(a, b) = 0$ and $f'_y(a, b) = 0$, where f'_x and f'_y are the **partial derivatives** of f. A **critical point** is where either $f'_x = f'_y = 0$ or at least one partial d. does not exist.

219. Calculate the first-order and second-order partial derivatives of the functions:

(a) $f(x, y) = xy$. $f_x = y$, $f_y = x$, $f_{xx} = f_{yy} = f_{xy} = 0$

(b) $z(x, y) = xe^{xy}$. $f_x = xye^{xy} + e^{xy}$, $f_y = x^2e^{xy}$,
$f_{xx} = xy^2e^{xy} + 2ye^{xy}$, $f_{yy} = x^3e^{xy}$, $f_{xy} = f_{yx} = x^2ye^{xy} + 2xe^{xy}$

(c) $z = x^2y + \ln(xy)$. $f_x = 2x\ln(xy) + x$, $f_y = \frac{x^2}{y}$,
$f_{xx} = 2\ln(xy) + 3$, $f_{yy} = \frac{-x^2}{y^2}$, $f_{xy} = f_{yx} = \frac{2x}{y}$

> The function $D(x, y) = f''_{xx}f''_{yy} - f''_{xy}f''_{yx}$ can be used to classify critical points.
> If $D > 0$ and $f''_{xx} > 0$ at a critical point, then that point is a local minimum.
> If $D > 0$ and $f''_{xx} < 0$ at a critical point, then that point is a local maximum.
> If $D < 0$ at a critical point then it is *not* a local extreme (it is a "saddle").
> If $D = 0$ then the point might be a local extreme but might not be.

220. Find the local extrema of $f(x, y) = x^2 + xy + y^2 - 2x - y$.

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$ give the system $\begin{cases} 2x + y - 2 = 0 \\ x + 2y - 1 = 0 \end{cases}$, which has only $(x, y) = (1, 0)$ as a solution.

$f_{xx}f_{yy} - (f_{xy})^2 = (2)(2) - (1)^2 = 3$ (for all x and y), so $\boxed{(1,0) \text{ is a local min}}$.

The value of f at this point is $\boxed{f(1, 0) = -1}$.

221. Find the local extrema of the function $z = x^3y^2(6 - x - y)$.

$\begin{cases} 18x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0 \\ 12x^3y - 2x^4y - 3x^3y^2 = 0 \end{cases}$ factors as $\begin{cases} x^2y^2(18 - 4x - 3y) = 0 \\ x^3y(12 - 2x - 3y) = 0 \end{cases}$.

When $x = 0$ or $y = 0$, the value of the function is $z = 0$.

The only other critical point is $(3, 2)$, a $\boxed{\text{local max}}$. The function value is $\boxed{f(3, 2) = 108}$.

To find extreme values of $f(x,y)$ with a condition/restriction, re-write the task as a single-variable extreme value task.
For extreme values of $f(x,y)$ in a polygonal domain, check the value of the function at all critical points inside the domain and at all vertices (corners) of the domain.

222. Find the maximum of the function (Cobb-Douglas production function)

$$u(x,y) = \sqrt{xy} = x^{1/2}y^{1/2}$$

describing the production value in the case that the parameters x and y satisfy the condition $7x + 3y = 84$.

Since $7x + 3y = 84$, we can change this to a function of only one variable:

$$u = f(x) = \sqrt{x \cdot \tfrac{84-7x}{3}} = \sqrt{28x - \tfrac{7}{3}x^2}.$$

Then $f' = \dfrac{28 - \frac{14}{3}x}{2\sqrt{28x - \frac{7}{3}x^2}}$ is zero exactly when $x = 6$, and f' is undefined (denominator is zero) when $x = 0$ or $x = 12$. Each of those has a corresponding y-value (e.g., for $x = 6$, it's $\frac{84-7(6)}{3} = 14$). Compare f-values at these points:

x	y	f
0	28	0
6	14	$\boxed{2\sqrt{21}\text{ max}}$
12	0	0

223. Determine the smallest and the largest value of $z = z(x,y)$ in the given region:

(a) $z = x^2 + 2xy - 4x + 8y$ in the region $D:\ 0 \le x \le 1,\ 0 \le y \le 2$,

System $\begin{cases} 2x + 2y - 4 = 0 \\ 2x + 8 = 0 \end{cases}$ leads to $(x,y) = (-4,6)$ as the only stationary point, but this is outside of the square D. Check the four sides of the square:

- When $x = 0$, we have $z = 8y$, so the min and max on this side are the endpoints (two of the corners of D).
- When $y = 0$, we have $z = x^2 - 4x$, leading to $(0,0)$ and $(4,0)$. The first of those is a vertex of D and the second is outside of D.
- When $x = 1$, $z = 10y - 3$, leading to $(x,y) = (1, \frac{3}{10})$.
- When $y = 2$, $z = x^2 + 16$, which has no zeros.

Compare function values at these points:

x	y	f
0	0	0
1	0	$\boxed{-3\text{ min}}$
1	2	$\boxed{17\text{ max}}$
0	2	16

(b) $z = x^3 + y^2 - 3x - 2y - 1$ in the region $D:\ x \ge 0,\ y \ge 0,\ x + y \le 1$.

The region is a triangle. $\begin{cases} 3x^2 - 3 = 0 \\ 2y - 2 = 0 \end{cases}$ gives two stationary points: $(1,1)$ and $(-1,1)$, but both are outside the triangle. Three sides:

- Left: When $x = 0$, we have $z = y^2 - 2y$, which has minimum when $y = 1$ (the point $(0, 1)$, which is already a vertex of the triangle).
- Bottom: When $y = 0$, we have $z = x^3 - 3x - 1$, which has $z'_x = 0$ when $x = \pm 1$ again.
- Diagonal: When $x + y = 1$, we have $y = 1 - x$, so

  $$z = x^3 + (1-x)^2 - 3x - 2(1-x) - 1 = x^3 + x^2 - 3x - 2.$$

  Setting the derivative $3x^2 + 2x - 3$ equal to 0 gives $x = \frac{-1\pm\sqrt{10}}{3}$. The point with $-$ is outside the triangle, but $x = \frac{-1+\sqrt{10}}{3}$ is important. Using $y = 1 - x$, we get $y = \frac{4-\sqrt{10}}{3}$ for this point.

Compare function values at the point on the diagonal and at the vertices:

x	y	f
0	0	$\boxed{-1 \text{ max}}$
1	0	-3
0	1	-2
$\frac{-1+\sqrt{10}}{3}$	$\frac{4-\sqrt{10}}{3}$	$\frac{-5}{27}(5+4\sqrt{10}) \approx \boxed{-3.268 \text{ min}}$

(c) $z = x^2 - xy + y^2$ in the region $D: \ |x| + |y| \leq 1$.

The region is a diamond (to see this, it might help to graph the four lines $x + y = 1$, $x - y = 1$, $-x + y = 1$, and $-x - y = 1$).

From $z'_x = 2x - y$ and $z'_y = 2y - x$, the only stationary point is $(0, 0)$. This is inside the diamond.

- Upper right side $x+y = 1$ gives $z = x^2 - x(1-x) + (1-x)^2 = 3x^2 - 3x + 1$, which leads to $6x - 3 = 0$, or $x = \frac{1}{2}$. This is the point $(\frac{1}{2}, \frac{1}{2})$.
- Upper left side $-x+y = 1$ gives $z = x^2 - x(1+x) + (1+x)^2 = x^2 + x + 1$, which leads to $2x + 1 = 0$, or $x = -\frac{1}{2}$. This is the point $(-\frac{1}{2}, \frac{1}{2})$.
- Lower left side... ...the point $(-\frac{1}{2}, -\frac{1}{2})$.
- Lower right side... ...the point $(\frac{1}{2}, -\frac{1}{2})$.

Compare function values at five points determined above and at the four vertices of the diamond:

x	y	f
0	0	$\boxed{0 \text{ min}}$
1/2	1/2	1/4
$-1/2$	1/2	3/4
$-1/2$	$-1/2$	1/4
1/2	$-1/2$	3/4
1	0	$\boxed{1 \text{ max}}$
0	1	$\boxed{1 \text{ max}}$
-1	0	$\boxed{1 \text{ max}}$
0	-1	$\boxed{1 \text{ max}}$

224. Find the distance of the point $A = (0, 3, 0)$ from the surface $y = zx$.

The closest point is $(-\sqrt{2}, 2, -\sqrt{2})$, a distance of $\boxed{\sqrt{5}}$ units away.

225. Write a positive number a as the sum of three positive numbers in such a way that the product of these three ingredients attains the maximal value. $\boxed{\frac{a}{3} + \frac{a}{3} + \frac{a}{3}}$

226. A cuboidal warehouse is supposed to have the volume $V = 64\,m^3$. One square meter of ceiling costs 20 zł, one square meter of the floor costs 40 zł and one square meter of the wall costs 30 zł. Determine the length a, width b and height c of the warehouse, minimizing the total cost. $\boxed{a = b = c = 4}$

227. The total annual income in the sale of two goods is expressed by the function

$$D(x, y) = 400x - 4x^2 + 1960y - 8y^2,$$

where x and y denote amounts of goods of respectively first and second type, sold per year. The production cost of x items of the first type and y items of the second type is: $K(x, y) = 100 + 2x^2 + 4y^2 + 2xy$. Determine the number of items of goods of the first and second type maximizing the annual profit. $\boxed{x = 20,\ y = 80}$

228. Suppose we have the budget of 4 000 000 PLN at our disposal. What should be the amounts spent on resources x and y, so as to minimize the production cost described by the function $f(x, y) = x^2 + y^2 - xy + 3$? $\boxed{x = 2,\ y = 2}$

229. Distribute the daily power production of 100 MWh between two power generating plants A and B in such a way so as to minimize the daily cost of fuel, given by the function

$$f(x, y) = 2(x - 1)^2 + (y - 3)^2,$$

where x is the use of the fuel at plant A and y is the use at plant B. Moreover, 1 tone of fuel supplies 5 MWh of energy at plant A and 1 tone of fuel supplies 3 MWh of energy at plant B. Give the daily cost of fuel use at both plants. $\boxed{x = 11,\ y = 15}$