Math for Management, Winter 2023

List 9

More functions of two variables

230. Give the partial derivative of

$$z = f(x, y) = xy^3 + x^5\, e^{xy} - 2^x$$

with respect to x, which is a new function with two inputs. We can write any of

$$f'_x(x, y) \qquad f_x(x, y) \qquad z'_x(x, y) \qquad z_x(x, y) \qquad f'_x \qquad f_x \qquad z'_x \qquad z_x$$

for this function.
It may help to think about $(ax + x^5\, e^{b\,x} - 2^x)'$, where a, b, c are constants.

231. Give the partial derivative of

$$z = f(x, y) = xy^3 + x^5\, e^{xy} - 2^x$$

with respect to y, which is a new function with two inputs. We can write any of

$$f'_y(x, y) \qquad f_y(x, y) \qquad z'_y(x, y) \qquad z_y(x, y) \qquad f'_y \qquad f_y \qquad z'_y \qquad z_y$$

for this function.
It may help to think about $(at^3 + b\,e^{c\,t} - d)'$, where a, b, c, d are constants.

232. (a) Calculate the partial derivative of $f(x, y) = y^x$ with respect to x, which is a function. We can write $f'_x(x, y)$ or $f_x(x, y)$ or f'_x or f_x for this.
   (b) Calculate the partial derivative of $f(x, y) = y^x$ with respect to y, which is a function. We can write $f'_y(x, y)$ or $f_y(x, y)$ or f'_y or f_y for this.
   (c) Calculate the partial derivative of $f(x, y) = y^x$ with respect to x at the point $(5, 2)$, which is a number. We can write $f'_x(5, 2)$ or $f_x(5, 2)$ for this.
   (d) Calculate the partial derivative of $f(x, y) = y^x$ with respect to y at the point $(5, 2)$, which is a number. We can write $f'_y(5, 2)$ or $f_y(5, 2)$ for this.

233. For the function $z = x^8 y^2$, calculate
   (a) z_x
   (b) $z_{xx} = (z_x)_x$
   (c) $z_{xy} = (z_x)_y$
   (d) z_y
   (e) $z_{yx} = (z_y)_x$
   (f) $z_{yy} = (z_y)_y$
   (g) $z_{xx}\, z_{yy} - z_{xy}\, z_{yx}$

The point $(x, y) = (a, b)$ is a **stationary point** of $f(x, y)$ if both $f'_x(a, b) = 0$ and $f'_y(a, b) = 0$, where f'_x and f'_y are partial derivatives of f. A **critical point** is where either $f'_x = f'_y = 0$ or at least one partial d. does not exist.

234. Find the stationary points of

$$f(x, y) = 2x^2(x - \tfrac{3}{2}y - 6) - 3e^{2\ln(y)}.$$

Hint: there are three.

235. Find all stationary points for each of the following functions.
   (a) $f(x, y) = e^{7x} - xy$
   (b) $z = x^3 + 8y^3 - 3xy$
   (c) $f = y\ln(x^2) + x$

The function $D(x,y) = f''_{xx}f''_{yy} - f''_{xy}f''_{yx}$ can be used to classify critical points.
If $D > 0$ and $f''_{xx} > 0$ at a critical point, then that point is a local minimum.
If $D > 0$ and $f''_{xx} < 0$ at a critical point, then that point is a local maximum.
If $D < 0$ at a critical point then it is *not* a local extreme (it is a "saddle").
If $D = 0$ then the point might be a local extreme but might not be.

236. Find and classify all the critical points of $f(x,y) = 2x^2(x - \frac{3}{2}y - 6) - 3e^{2\ln(y)}$.

To find extreme values of $f(x,y)$ with a condition/restriction, re-write the task as a single-variable extreme value task (see List 6 for those kinds of tasks).
For extreme values of $f(x,y)$ in a polygonal domain, check the value of the function at all critical points inside the domain, at all vertices (corners) of the domain, and use one-variable tasks along each side.

237. Find the smallest and largest values of

$$f(x,y) = 9x^2 - 6x - y^3 - y^2 + 9$$

on the filled square $0 \leq x \leq 1$, $0 \leq y \leq 1$ by following these steps:

(a) Find the critical points of $f(x,y)$. Ignore any that do not satisfy $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

(b) Find single-variable-critical-points on the boundary of the square by looking at each side separately.

i. Bottom: when $y = 0$, the function is $f = 9x^2 - 6x - (0)^3 - (0)^2 + 9$, so use Analysis 1 tools to find when $g(x) = 9x^2 - 6x + 9$ has $g'(x) = 0$. Remember that—for part (c) below—these are all points $(\_\_, 0)$.

ii. Top: when $y = 1$, the function is $f = 9x^2 - 6x - (1)^3 - (1)^2 + 9$.

iii. Left: when $x = 0$, the function is $f = 9(0)^2 - 6(0) - y^3 - y^2 + 9$, so use Analysis 1 tools to find when $g(y) = 9 - y^2 - y^3$ has $g'(y) = 0$.

iv. Right: when $x = 1$, the function is $f = 9(1)^2 - 6(1) - y^3 - y^2 + 9$.

(c) Compute the value of f at all points from steps (a) and (b). The largest f-value is the maximum and the most negative f-value is the minimum.

238. Find the extreme values of

$$z = (x-3)^2 + (y+1)^2 - 2(y + 5 - 2x)$$

on the solid triangular region with vertices $(2,0)$, $(0,2)$, and $(0,-2)$.