Analysis 2

5 March 2024

Warm-up: Calculate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\sin(x^4 + 5^3) + 5x^2\right]$.

Topics

The first half of year will focus on “multi-variable calculus”.

- path integrals
- partial derivatives and directional derivatives
- local extremes of multi-variable functions
- double integrals

The second half of the year will focus on “differential equations”.

- vocab for ODEs and IVPs in general
- separable ODEs
- first-order linear ODEs
- higher-order linear ODEs with constant coefficients

Calculations vs. Ideas

In Analysis 1, I often tried to focus on *ideas first*.

- For example, I talked about finding min / max using critical points before we learned the Product Rule.

For some topics this year, I'm going to do *calculations first*.

- We will do calculations like

$$\frac{\partial}{\partial y}\left[x^5y^9\right] = 9x^5y^8 \qquad \text{and} \qquad \int_0^1\int_3^5 xy\,\mathrm{d}x\mathrm{d}y = 4$$

  without worrying—at first—about what they mean or what problems could be solved using those calculations.

- That discussion will come later.

Grades

Quizzes worth 5 points each

- 6 total. Lowest is dropped.
- There will also be “bonus points” from Portal, but the maximum score for each quiz is still 5 / 5. (Details on next slide.)

Exams worth 15 points each

- Exam 2 might be last day of class (with second attempt during the University’s exam period). Will be decided later.

Participation worth 5 points

Grades

Quizzes — details

- There will be in-class quizzes approximately every two weeks. **The maximum score on each quiz is 5 points.**
- On most weekends, ePortal will have a single “bonus” task.
    - If you answer the ePortal task correctly, you earn an extra point on the quiz, so you could make a small mistake in class and still get full credit.
    - If you answer the ePortal task incorrectly, you can still get 5 / 5 from the in-class quiz!

Realistically, I know students can cheat on online tasks. If you do, you might earn the 1 point, but you will not understand the material as well, so you will probably not do well on the in-class quiz!

Grades

There are 6×5 + 15+15 + 5 = **60** total possible points.

Points	[0, 30)	[30, 36)	[36, 42)	[42, 48)	[48, 54)	[54, 60]
Grade	2.0	3.0*	3.5*	4.0	4.5	5.0

**** Passing also requires at least 12 points from each "half" of the course.***

You can work together on task lists (which are not graded).

Quizzes and exams should be individual.

- Cheating on in-class quiz → quiz grade 0.
- Cheating on exam → course grade 2.0.

More than 5 unexcused absences after this week → course grade 2.0.

Accessibility

Department of Accessibility and Support for People with Disabilities (DDO)

- Office: C-13 rooms 109 and 107
- Telephone: 71 320 43 20
- Website: https://ddo.pwr.edu.pl/
- Email: pomoc.n@pwr.edu.pl

If you need any kind of accommodation, please write me an email.
I am happy to help.

Scalars and vectors

Of course math has numbers:

- 5
- $\frac{1}{3}$
- π^2
- $\ln(3)$

These are also called **scalars** sometimes.

You have hopefully also seen vectors

- $[5,0,2]$
- $5\hat{\imath} + 2\hat{k}$

these are the same

- $3\hat{\imath} - 4\hat{\jmath}$
- $\begin{bmatrix} -6 \\ 2/9 \end{bmatrix}$

and matrices

- $\begin{bmatrix} 2 & 0 \\ -1 & 9 \end{bmatrix}$
- $\begin{pmatrix} 2 & 0 \\ -1 & 9 \end{pmatrix}$

in other classes.

Scalars and vectors

We will only use two important vector ideas/calculations in this class:

1) The **length** or magnitude of the vector $\vec{v} = [a, b, c]$ is written as $\left|\vec{v}\right|$ and is calculated as

$$\left|\vec{v}\right| = \sqrt{a^2 + b^2 + c^2}.$$

- A unit vector has length 1.
- If $s > 0$ is a scalar, then $\left|s\vec{v}\right| = s\left|\vec{v}\right|$.

2) The **dot product** or scalar product of two vectors $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$ is written as $\vec{a} \cdot \vec{b}$ and has two formulas:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3, \qquad \vec{a} \cdot \vec{b} = \left|\vec{a}\right| \left|\vec{b}\right| \cos(\theta),$$

where θ is the angle between the two vectors.

Task 1: If $\overrightarrow{w} = -5\hat{\imath} + 2\hat{\jmath}$, what is $\left|\overrightarrow{w}\right|$?

Answer: $\sqrt{29}$

Task 2: If $\vec{v} = a\hat{\imath} + b\hat{\jmath}$, what is $\left|\vec{v}\right|$?

Answer: $\sqrt{a^2 + b^2}$

Task 3: If $\vec{q} = 4n\hat{\imath} + p^3\hat{\jmath}$, what is $\left|\vec{q}\right|$?

Answer: $\sqrt{16n^2 + p^6}$

The last task might look strange because has several variables, but it's really exactly the same formula (the Pythagorean formula) you would use to find the longest side of a right triangle!

Functions

The functions

- $f(x) = x^3$
- $f(x) = \sin(x)$
- $g(x) = 2e^{9x}$
- $h(t) = t^8 + \ln(1 + \cos(1/t))$

all have <u>*one*</u> *input* and *one output*. (We can write $f : \mathbb{R} \to \mathbb{R}$ for this.)

We can graph these kinds of functions on a 2D drawing, and we can find the derivative, the tangent line at a point, the indefinite or definite integral, etc.

Now we will start looking at functions with <u>*multiple*</u> *inputs*, so $f : \mathbb{R}^n \to \mathbb{R}$.

	a scalar (number) as output	a vector (or multiple numbers) as output
a scalar (number) as input	$f(x)$ $x(t)$ $P(x)$	$\vec{r}(t)$
a vector (or multiple numbers) as input	$f(x, y, z)$ $T(x, t)$	$\overrightarrow{F}(x, y, z)$

	a scalar (number) as output	a vector (or multiple numbers) as output
a scalar (number) as input		"vector function" (also used for parametric curves)
a vector (or multiple numbers) as input	"scalar function" or "scalar field"	"vector field"

A function $\mathbb{R}^n \to \mathbb{R}$ is called a **scalar function**. For functions $\mathbb{R}^2 \to \mathbb{R}$ we often write

$$f(x, y)$$

although the letters do not have to be "f" and "x" and "y".

These often come from physical descriptions:

- $r(x, y) = \sqrt{x^2 + y^2}$ distance from (x,y) to the origin
- $P(\ell, w) = 2\ell + 2w$ perimeter of length l, width w rectangle
- $K(m, v) = \frac{1}{2}mv^2$ kinetic energy for mass m and speed v

We can have even more inputs, such as

- $T(x, y, z, t)$ temperature at different places and times

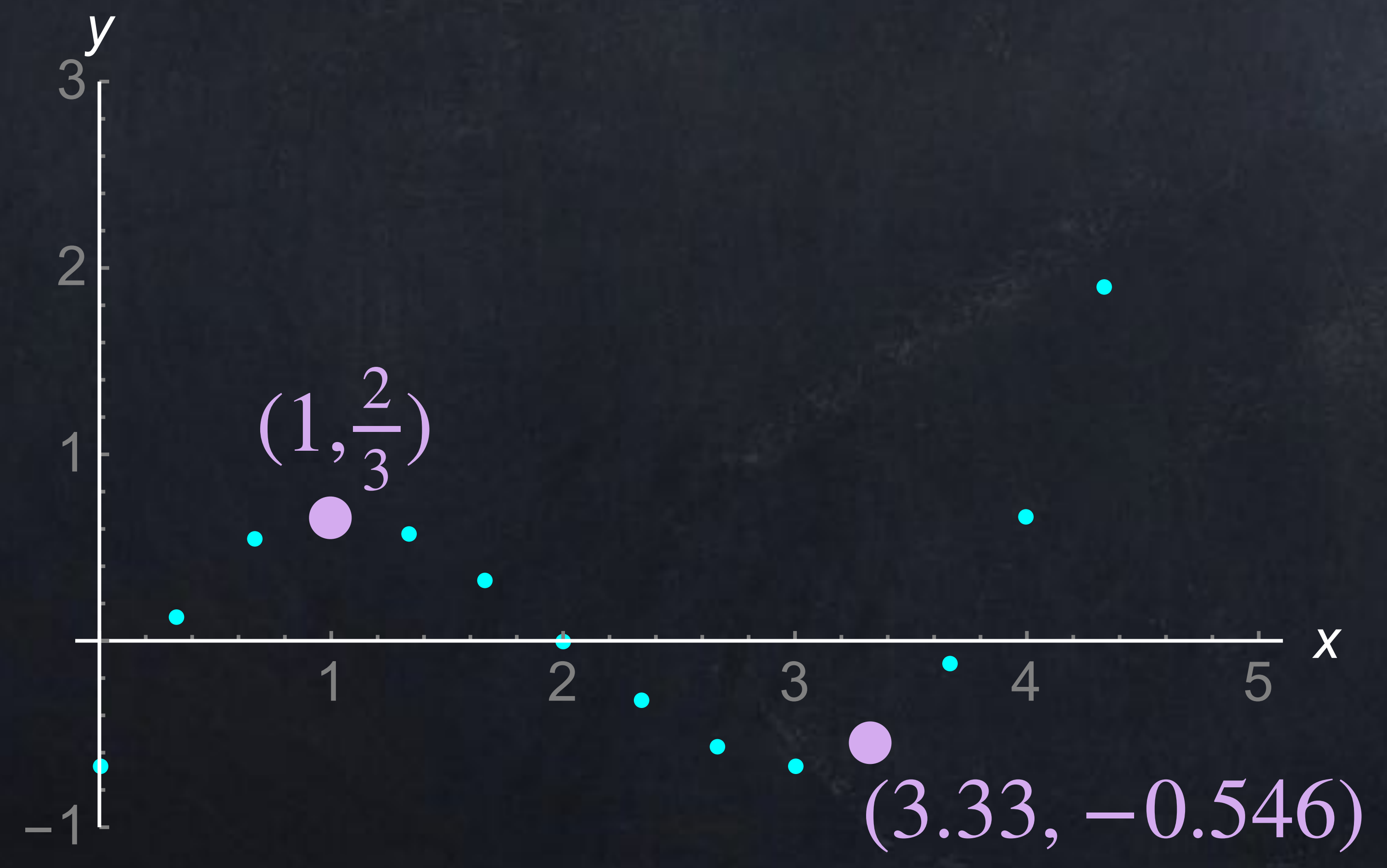
Graphs of functions

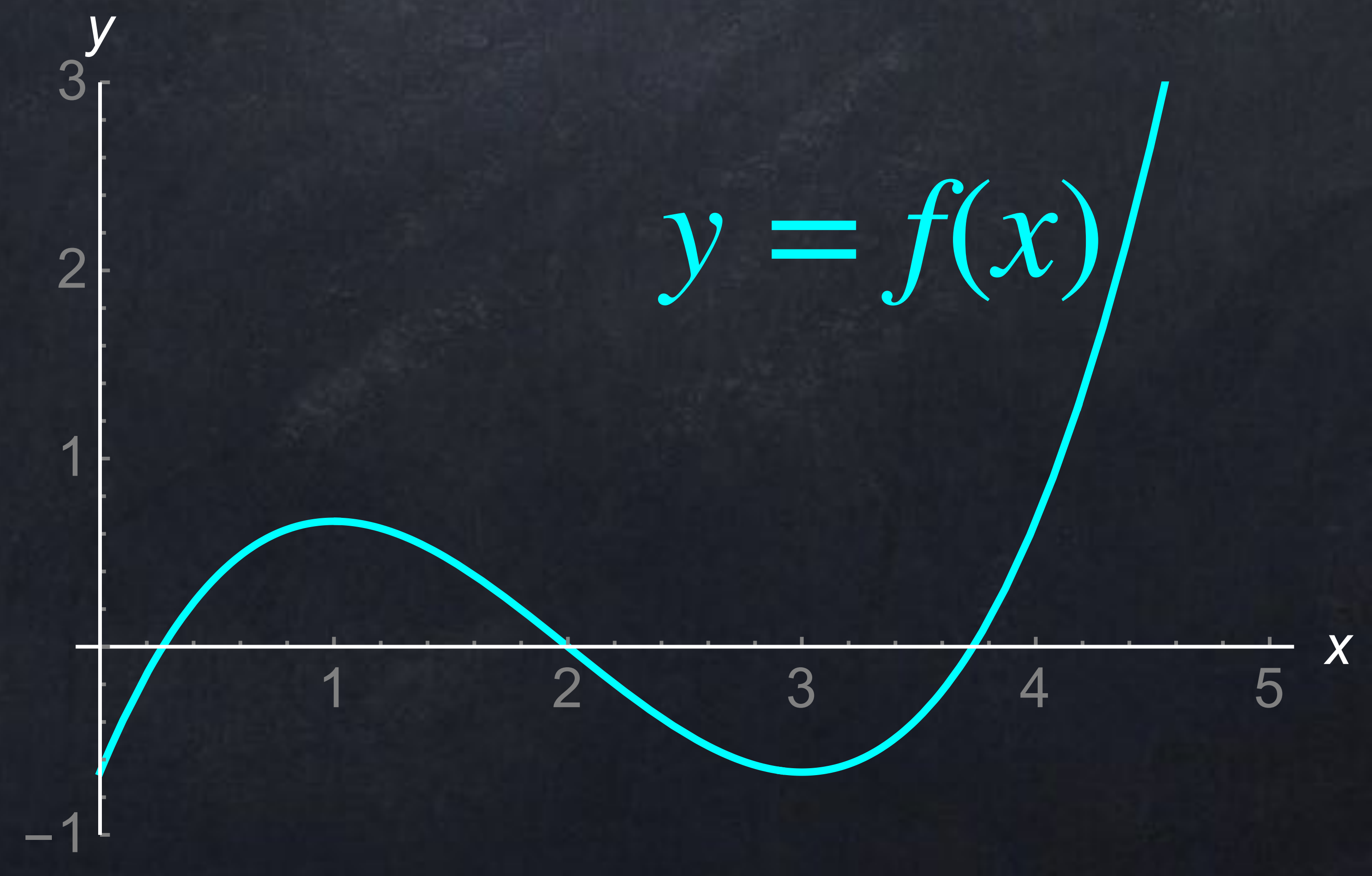
The graph of a function $f(x)$ is the set of all points (x, y) in 2D for which

$$y = f(x).$$

We draw this on a plot with the input on the horizontal axis and the output on the vertical axis.

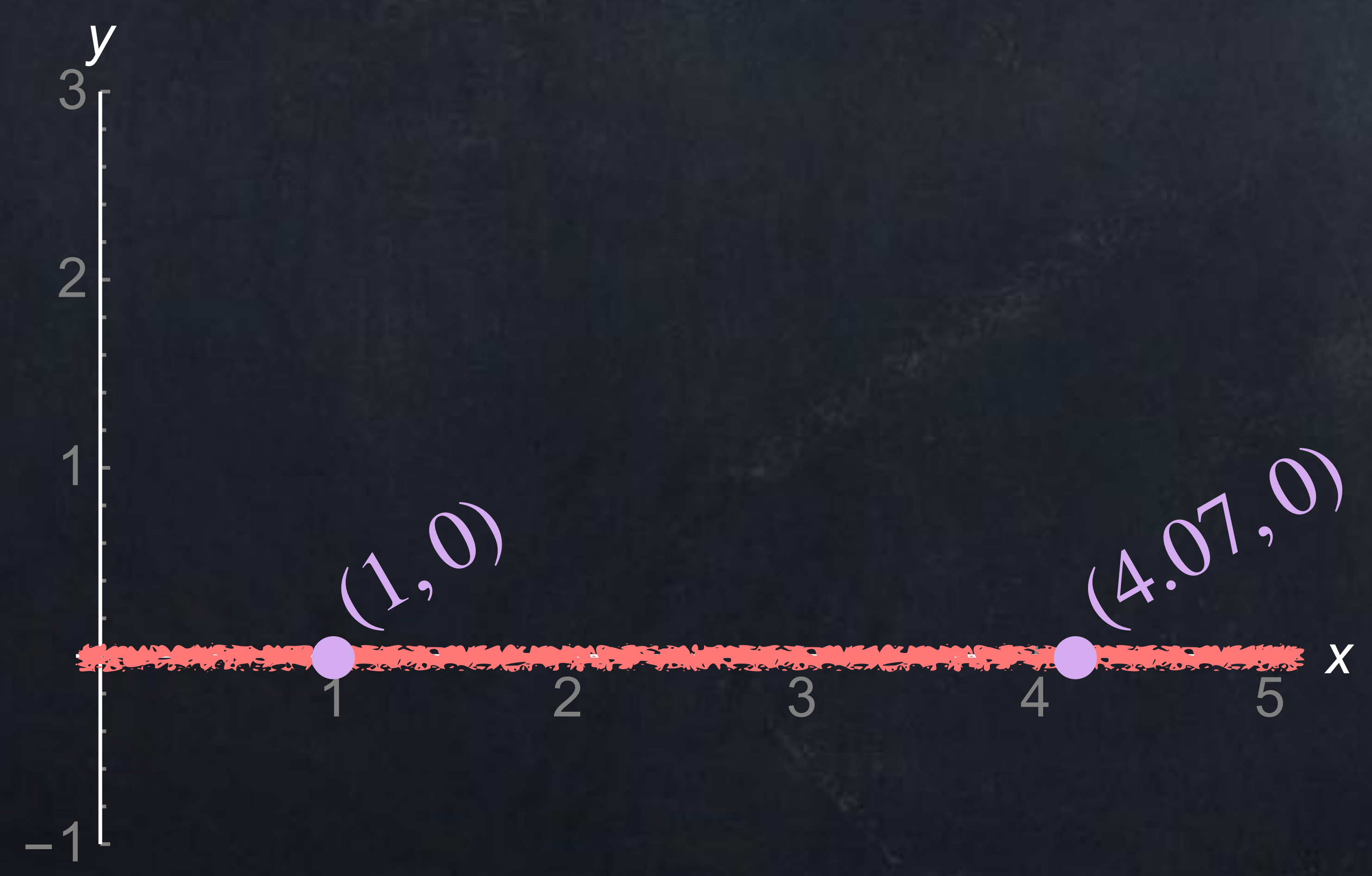
$$f(x) = \frac{1}{3}x^3 - 2x^2 + 3x - \frac{2}{3}$$



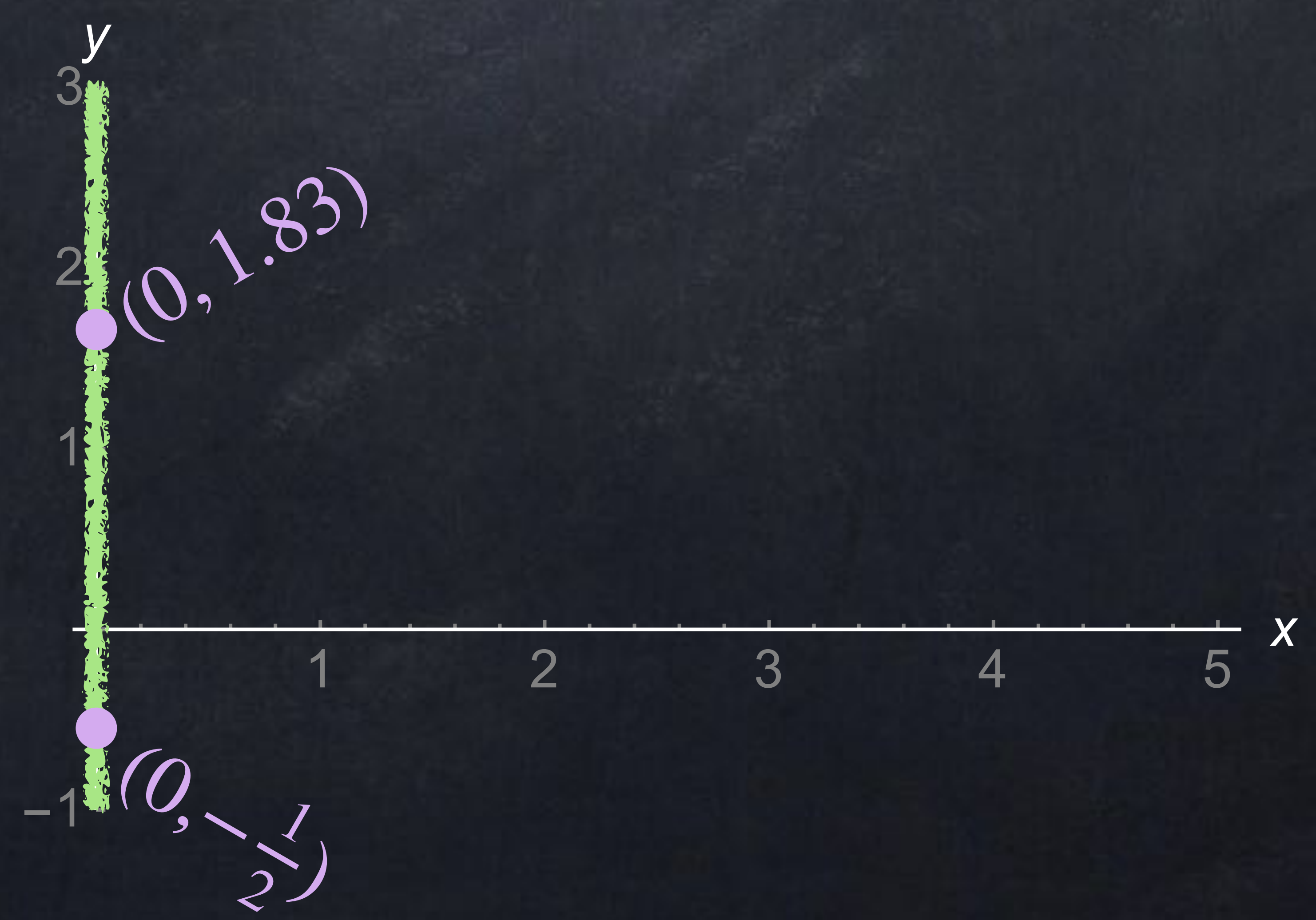


Graphs of functions

The x-axis has all points with y=0.



The y-axis has all points with x=0.

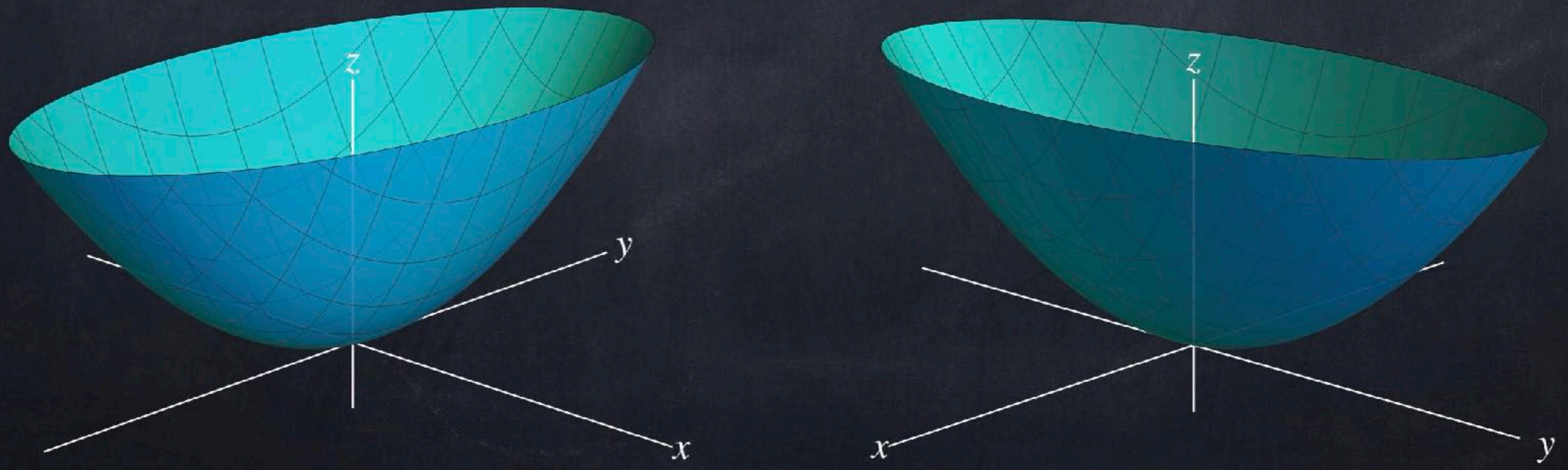


For a function with *two* inputs, we need a *3*D picture for the graph.

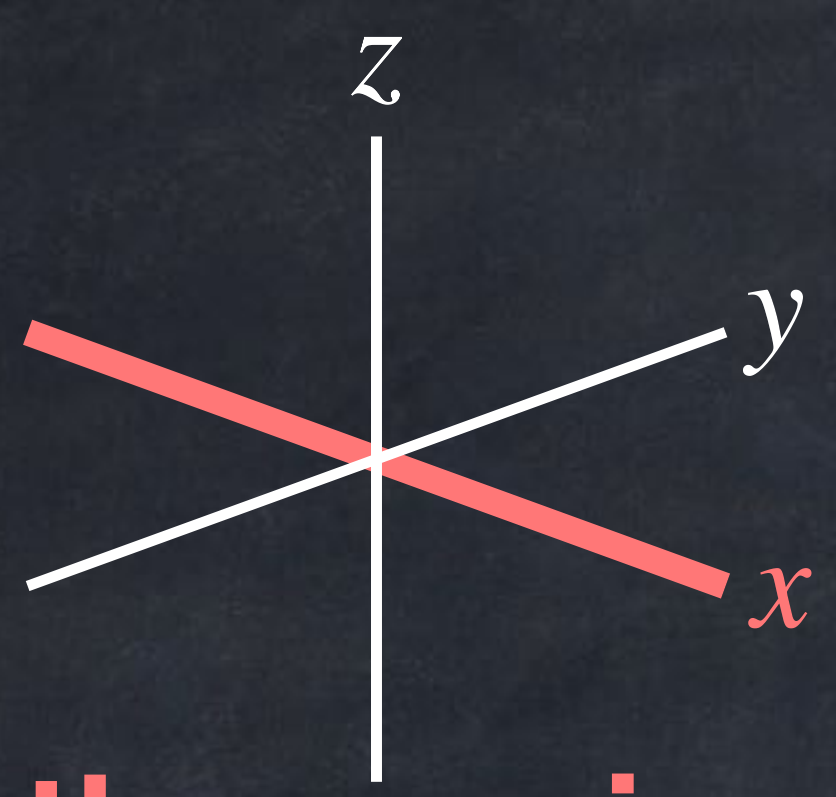
The graph of $f(x, y)$ is the set of points (x, y, z) in 3D for which

$$z = f(x, y).$$

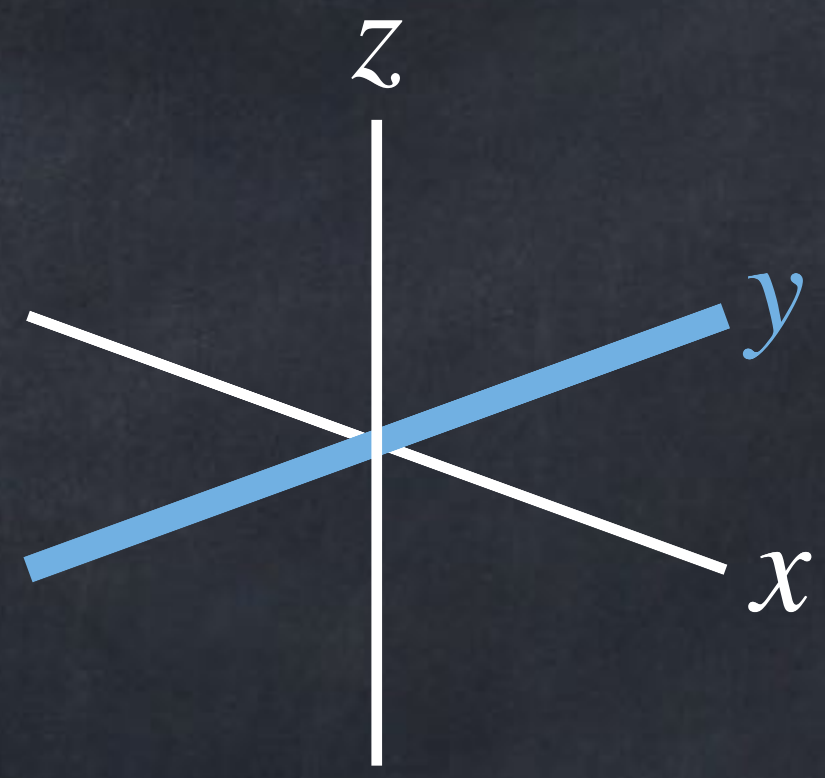
Example: The graph of $f(x, y) = 4x^2 + y^2$, drawn from two perspectives:



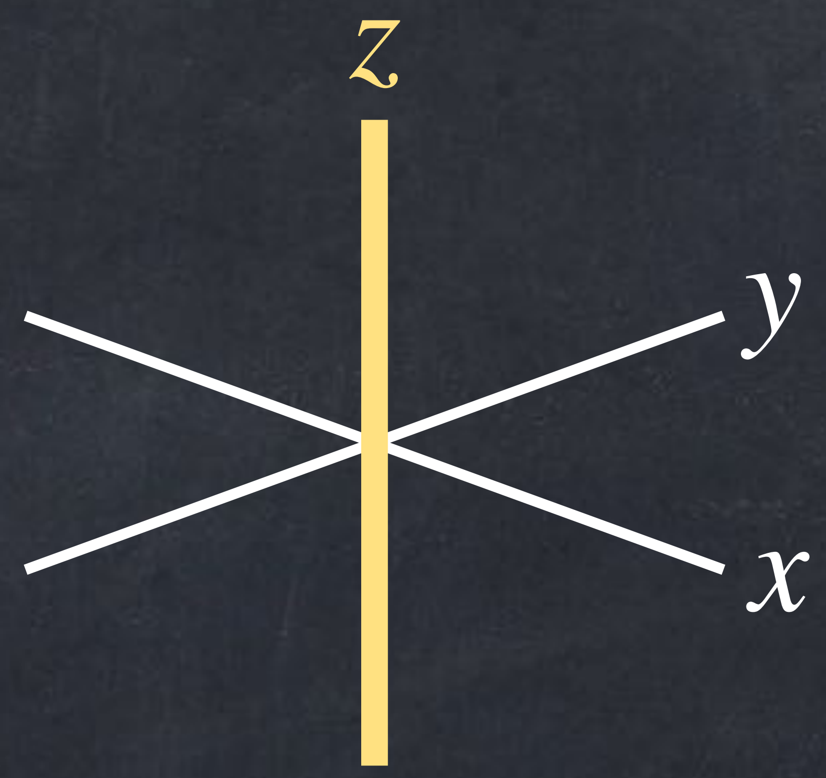
The "coordinate axes" and the "coordinate planes" in 3D are these:



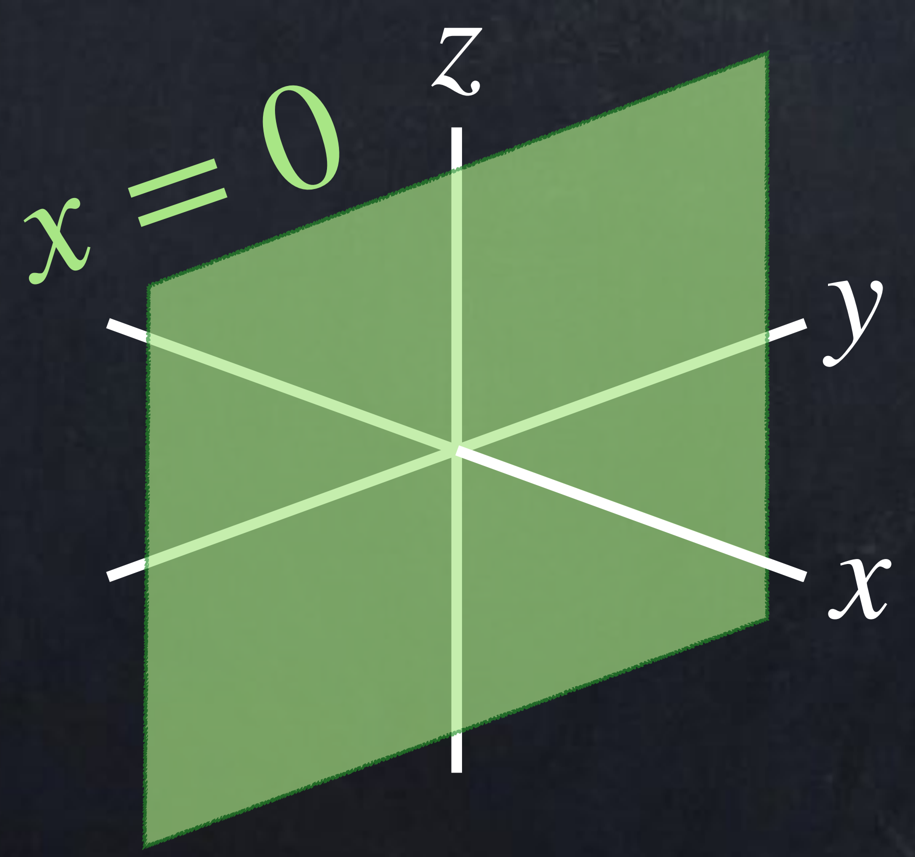
the x-axis



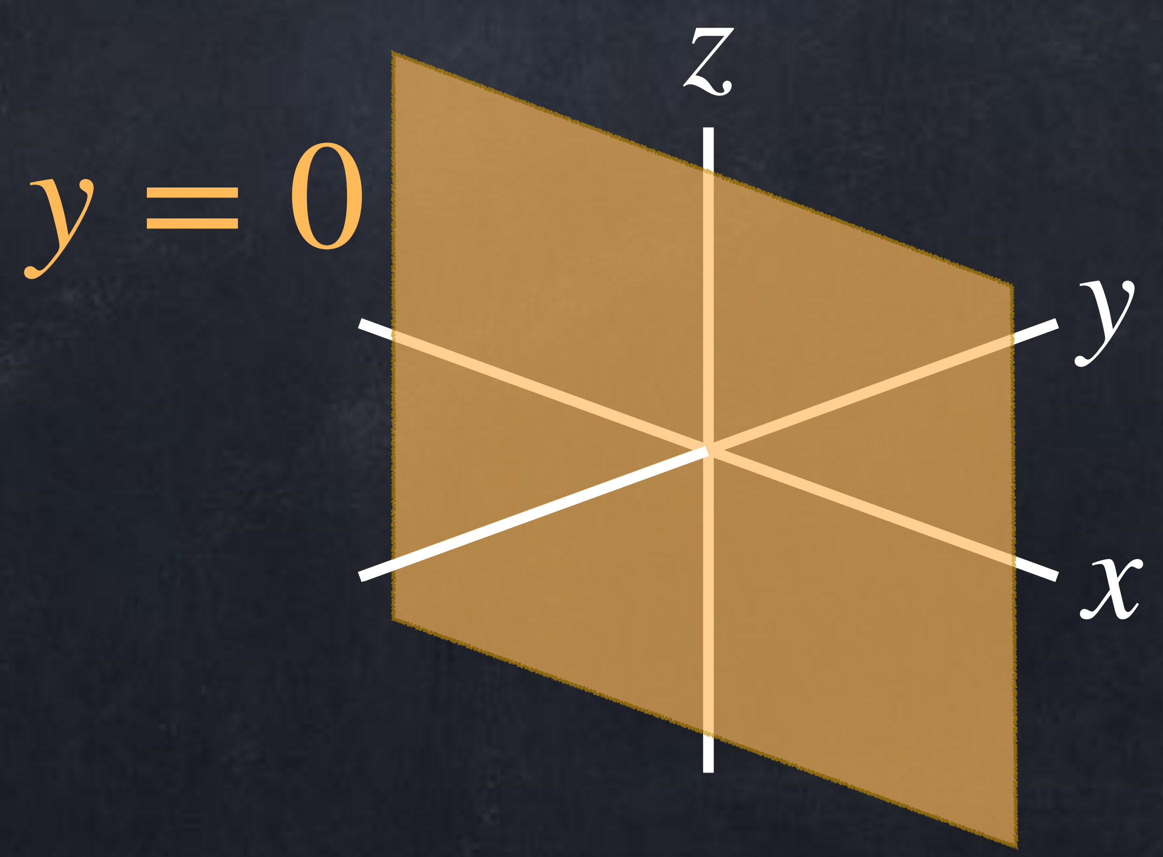
the y-axis



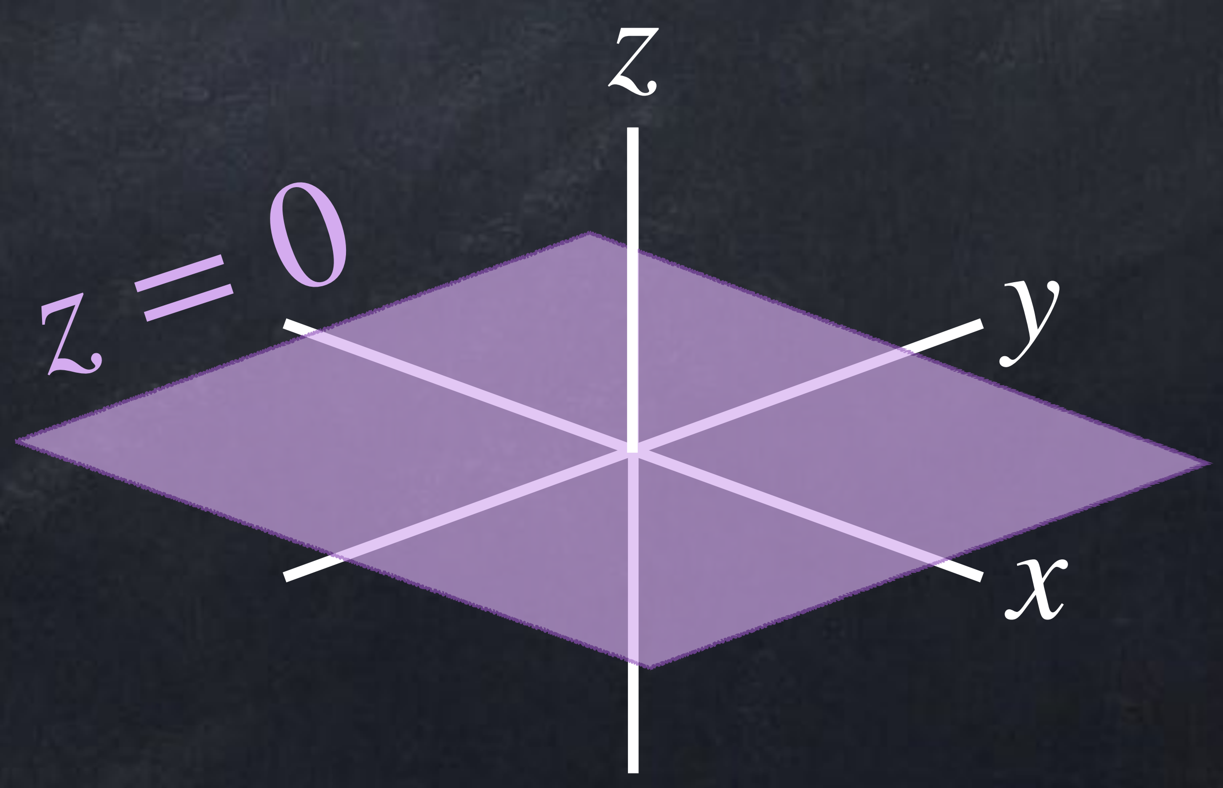
the z-axis



the yz-plane



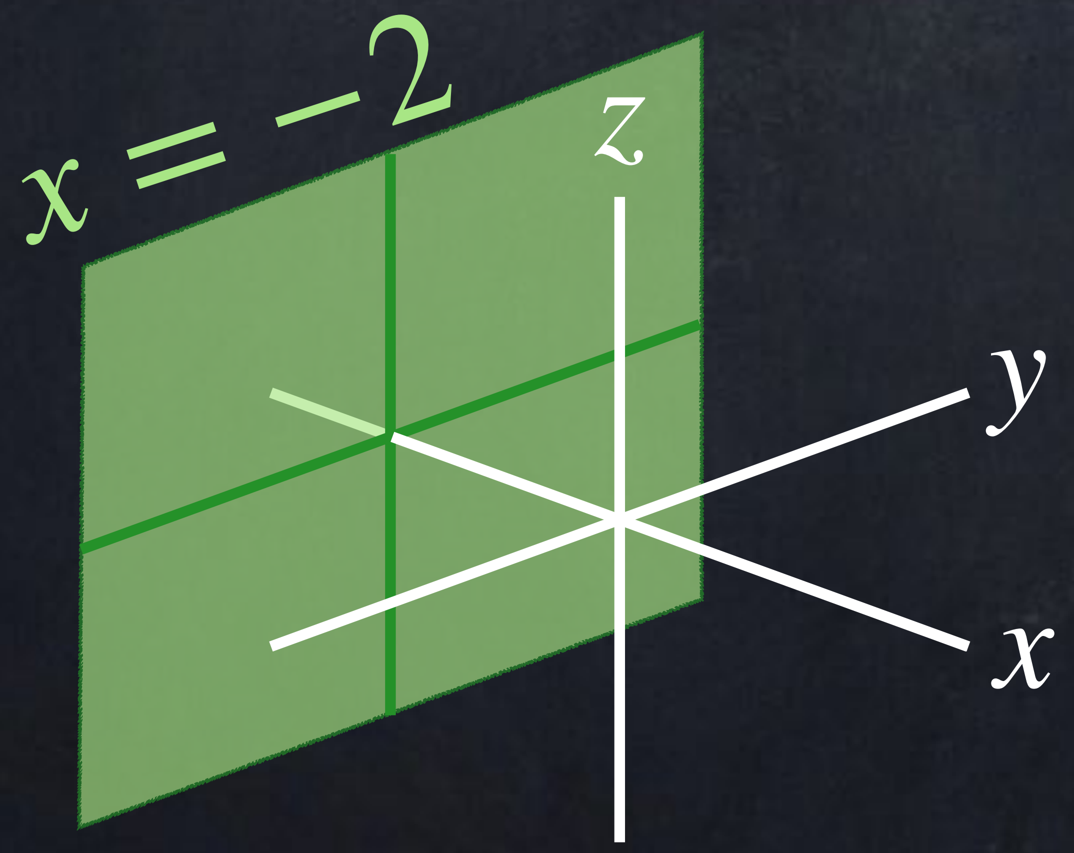
the xz-plane

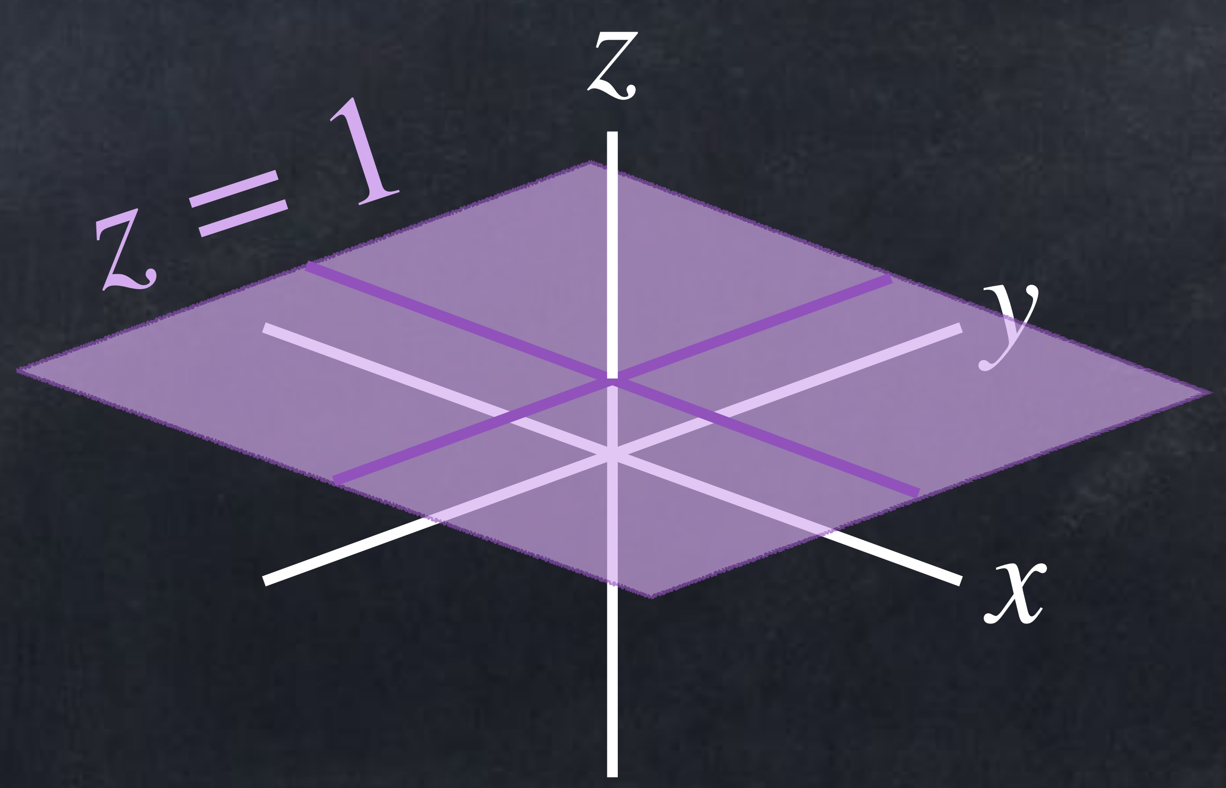


the xy-plane

In general, equations of planes can look like $Ax + By + Cz = D$ (you might have seen this in your Linear Algebra class). A plane that is parallel to a coordinate plane has a much more simple formula. Examples:

- The plane $x = -2$ is parallel to the yz-plane (which is $x = 0$).
- The plane $z = 1$ is parallel to the xy-plane (which is $z = 0$).





Curves

A curve in 2D space might be described as a graph, like $y = x^3 - x$, but some curves are easier to describe with "parametric equations". For example, the parametric equations

- $x(t) = \cos t, \quad y(t) = \sin t$

with $-\frac{3\pi}{4} \leq t \leq \frac{\pi}{4}$ draw part of the circle $x^2 + y^2 = 1$.

In 3D, curves are almost always described by parametric equations. Example:

- $x(t) = 8\cos t, \quad y(t) = 3, \quad z(t) = 8\sin t$

describes a circle that is in a vertical plane (the plane $y = 3$).

Instead of listing separate equations for x and y, we can also write a set of parametric equations as a single vector equation. For example,

- $x(t) = \cos t, \quad y(t) = e^{10t}$

is the same as

- $\vec{r}(t) = (\cos t)\hat{\imath} + e^{10t}\,\hat{\jmath}.$

NOTE: We use $\vec{r}$ to mean either [x, y] or [x, y, z] depending on context.

We can do algebra with this function. For example,

- $\left|\vec{r}(t)\right| = \sqrt{(\cos t)^2 + e^{20t}}.$

We can also do analysis with this function! For example,

- $\vec{r}\,'(t) = (-\sin t)\hat{\imath} + 10e^{10t}\,\hat{\jmath}.$

Analysis 1 integrals

For $f(x)$, we had indefinite integrals, like $\int x \, \mathrm{d}x = \frac{1}{2}x^2 + C$, and we had definite integrals, like $\int_0^6 x \, \mathrm{d}x = 18$.

Although we never did this, it's also possible to write

$$\int_I x \, \mathrm{d}x = 18, \qquad \text{where } I = [0,6] \text{ or } I = \{x : 0 \leq x \leq 6\}.$$

using a single letter for an interval. This is "the integral of x 'over' the interval I ".

For $f : \mathbb{R}^n \to \mathbb{R}$ we will only have definite integrals.

Path integrals

If the letter C is used for a curve, then the **path integral** of the function $f(x, y)$ over the curve C is written as

$$\int_C f(x, y)\,\mathrm{d}s.$$

This can also be called “line integral” or “curve integral” or “curvilinear integral”.

Questions:

- What does this physically mean? → Next week
- How do we calculate this? → Today

Answer: $\int_a^b f(\vec{r}(t))\left|\vec{r}\,'(t)\right|\mathrm{d}t$ if the curve C is described by $\vec{r} : [a, b] \to \mathbb{R}^n$.

Example 1: Calculate

$$\int_C xe^{2y}\,\mathrm{d}s$$

where C is the curve with parametric equations $x(t) = 3\cos t$, $y(t) = 3\sin t$ for $0 \leq t \leq \frac{\pi}{2}$.

$\vec{r} = [3\cos t, 3\sin t]$, so $\vec{r}' = [-3\sin t, 3\cos t]$ and $|\vec{r}'|$ simplifies to 3.

The integral is $\int_0^{\pi/2} (3\cos t)e^{2(3\sin t)}\, 3\, dt = \frac{3}{2}e^{6\sin t}\Big|_{t=0}^{t=\pi/2} = \boxed{\frac{3}{2}(e^6-1)}.$

Path integral calculation: $\int_C f\,\mathrm{d}s = \int_a^b f(\vec{r}(t))\left|\vec{r}'(t)\right|\mathrm{d}t$

Example 2: Calculate the path integral of

$$f(x, y) = \sqrt{1 + 4x^2}$$

over the curve given by

$$x = t, \quad y = t^2, \quad 0 \leq t \leq 1.$$

Answer: $\dfrac{7}{3}$

Path integral calculation: $\displaystyle\int_C f\,\mathrm{d}s = \int_a^b f(\vec{r}(t))\left|\vec{r}'(t)\right|\mathrm{d}t$

DERIVATIVES

For a function of one variable, the derivative of $f(x)$ can be written as any of

$$f'(x) \qquad f' \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left[f\right] \qquad \frac{\mathrm{d}f}{\mathrm{d}x}$$

and is a new function. There is also the derivative at a point, which is just a number (the slope of the tangent line to $y = f(x)$ at the x-value).

The derivative is originally defined as a limit, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, but we don't use that formula once we learn rules like

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^p\right] = px^{p-1}, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left[\sin(x)\right] = \cos(x), \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(x)\right] = \frac{1}{x}.$$

if p is constant

DERIVATIVES

For a function $f(x, y)$ we can change x or change y (or both at once—more on that later), so we have multiple ways to take derivatives.

The **partial derivative of $f(x, y)$ with respect to x** can be written as any of

$$f'_x(x, y) \qquad f'_x \qquad D_x f \qquad \partial_x f \qquad \frac{\partial f}{\partial x}$$

and is what you get if you *think of every letter other than x as a constant*.

Like with $f'(x)$ and $f'(a)$, we also have the **partial derivative of f with respect to x at the point (a, b)**, which is a single number; we write this as $f'_x(a, b)$.

To calculate the partial derivative (function) of $f(x, y)$ with respect to x, we just pretend that any variable other than x is a constant.

$$\frac{\partial}{\partial x}\left[\sin(x^4 + y^3) + x^2 y\right] = ?$$

It might help to think of functions with one variable and a similar format….

To calculate the partial derivative (function) of $f(x, y)$ with respect to x, we just pretend that any variable other than x is a constant.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin(x^4 + 5^3) + 5x^2\right] = \cos(x^4 + 5^3) \cdot 4x^3 + 10x$$

$$\frac{\partial}{\partial x}\left[\sin(x^4 + k^3) + kx^2\right] = \cos(x^4 + k^3) \cdot 4x^3 + 2\,kx$$

$$\frac{\partial}{\partial x}\left[\sin(x^4 + y^3) + x^2y\right] = \cos(x^4 + y^3) \cdot 4x^3 + 2\,xy$$

To calculate the partial derivative (function) of $f(x, y)$ with respect to x, we just pretend that any variable other than x is a constant.

$$\left(\sin(x^4 + 5^3) + 5x^2\right)' = \cos(x^4 + 5^3) \cdot 4x^3 + 10x$$

$$\left(\sin(x^4 + k^3) + kx^2\right)'_x = \cos(x^4 + k^3) \cdot 4x^3 + 2\,kx$$

$$\left(\sin(x^4 + y^3) + x^2y\right)'_x = \cos(x^4 + y^3) \cdot 4x^3 + 2\,xy$$

There is also **the partial derivative of $f(x, y)$ with respect to y**, in which we think of every letter other than y as a constant (that includes x!).

$$\left(\sin(8^4 + t^3) + 8^2 t\right)' = \cos(8^4 + t^3) \cdot 3t^2 + 8^2$$

$$\left(\sin(8^4 + y^3) + 8^2 y\right)'_y = \cos(8^4 + y^3) \cdot 3y^2 + 8^2$$

$$\left(\sin(x^4 + y^3) + x^2 y\right)'_y = \cos(x^4 + y^3) \cdot 3y^2 + x^2$$

There is also **the partial derivative of $f(x, y)$ with respect to y**, in which we think of every letter other than y as a constant (that includes x!).

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\sin(8^4 + t^3) + 8^2 t\right] = \cos(8^4 + t^3) \cdot 3t^2 + 8^2$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[\sin(8^4 + y^3) + 8^2 y\right] = \cos(8^4 + y^3) \cdot 3y^2 + 8^2$$

$$\frac{\partial}{\partial y}\left[\sin(x^4 + y^3) + x^2 y\right] = \cos(x^4 + y^3) \cdot 3y^2 + x^2$$

Task 1: Calculate $\dfrac{\partial}{\partial x}\left[y^2 \sin(x)\right]$. This is f_x' for the function $f(x, y) = y^2 \sin(x)$.

You can think about $\dfrac{d}{dx}(k^2 \sin x)$ if it helps.

Answer: $y^2\cos(x)$.

Task 2: Calculate $\dfrac{\partial}{\partial y}\left[y^2 \sin(x)\right]$. This is f_y' for the function $f(x, y) = y^2 \sin(x)$.

You can think about $\dfrac{d}{dt}(t^2 \sin 1)$ if it helps.

Answer: $2y\sin(x)$.