Analysis 2

Tuesday, 14 May 2024

$2e^{2t}\sin(t) + e^{2t}\cos(t)$

Warm-up: (a) What is the derivative of $e^{2t}\sin(t)$?

(b) What is the derivative of $e^{-t}g(t)$?

$-e^{-t}g(t) + e^{-t}g'(t)$

Topics to review

Partial fractions.

- Example 1: $\dfrac{2x-3}{x^2-5x-6} = ?$
- Example 2: $\dfrac{25x}{(x^2+4)(x+1)^2} = ?$

Integration by parts: $\int u dv = uv - \int v du.$

- Example: $\int t\, e^t dt = ?$

Topics to review

Partial fractions.

- Example 1: $\dfrac{2x-3}{x^2-5x-6} = \dfrac{9/7}{x-6} + \dfrac{5/7}{x+1}$
- Example 2: $\dfrac{25x}{(x^2+4)(x+1)^2} = \dfrac{-3x+8}{x^2+4} + \dfrac{3}{x+1} + \dfrac{-5}{(x+1)^2}$

When the denominator has a repeated root, you need separate partial fractions for each power.

Integration by parts: $\displaystyle\int u\,dv = uv - \int v\,du.$

- Example: $\displaystyle\int t\,e^t dt$. Choosing $u = t$, $v = e^t$, $du = dt$, $dv = e^t dt$ gives

$$\int t\,e^t dt = t\,e^t - \int e^t dt = \boxed{t\,e^t - e^t + C.}$$

Separable ODE

Last time

A **separable first-order ODE** for $y(x)$ can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x) \cdot g(y).$$

Its solution can usually be found using

$$\int \frac{1}{g(y)} \mathrm{d}y = \int f(x)\, \mathrm{d}x.$$

Whether these integrals are easy or not depends on g and f.

- Any direct first-order ODE is separable: $g(y) = 1$ gives $y' = f(x)$.
- Any autonomous first-order ODE is separable: $f(x) = 1$ gives $y' = g(y)$.

Linear ODE

In standard form, a **first-order linear ODE** for $y(t)$ is

$$y' + a(t)\,y = f(t).$$

In standard form, a **second-order linear ODE** for $y(t)$ is

$$y'' + a(t)\,y' + b(t)\,y = f(t).$$

In standard form, a **third-order linear ODE** for $y(t)$ is

$$y''' + a(t)\,y'' + b(t)\,y' + c(t)\,y = f(t).$$

- We could write $y(t)$, $y'(t)$, etc., instead of just y, y'.
  Skipping some of the "(t)"s just makes the equation easier to read.
- We can use other letters: $x' + a(t)\,x = b(t)$ is a linear ODE for $x(t)$.

A first-order linear ODE for $y(x)$ can be written in the form

$$y' = P(x)\, y + Q(x)$$

for some functions $P(x)$ and $Q(x)$.

$$y' = xy + x^3$$ is **linear**

A first-order linear ODE for $y(x)$ can be written in the form

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$$y' = xe^y + 5$$ is **not linear**

A first-order linear ODE for $y(x)$ can be written in the form

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for some functions $P(x)$ and $Q(x)$.

$$y' = e^x\,y$$

is

linear

and

separable

A first-order linear ODE for $y(x)$ can be written in the form

$$y' = P(x)\,y + Q(x)$$

for some functions $P(x)$ and $Q(x)$.

$$x' = P(t)\,x + Q(t)$$

$$x' = 3t^2x + 4$$ is **linear**

Linear ODE

In standard form, a **first-order linear ODE** for $y(t)$ is

$$y'(t) + a(t)\,y(t) = f(t).$$

- If both $a(t)$ and $f(t)$ are constant functions, then we have

  $$y'(t) + a\,y(t) = b,$$

  which is separable (in fact, autonomous), so we can already solve this.

- A **homogeneous** first-order linear ODE looks like

  $$y'(t) + a(t)\,y(t) = 0.$$

  This is also separable, so we can solve this too.

  - The solution is $y = Ce^{-A(t)}$, where $A'(t) = a(t)$.

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
13 May	14 May (today)	15 Problem Session Quiz 5	16	17	18	19
20	21 Lecture	22	23	24	25	26
27	28 Lecture	29	30	31	1 June	2
3	4 Lecture	5 Problem Session Quiz 6 ?	6	7	8	9

First-order linear

There are four tools we can (sometimes) use to solve first-order linear ODEs.

- 🤔 Laplace transforms
- variation of parameters
- integrating factor (next week)
- big formula (next week)

$$y' + a(t)\,y = f(t)$$

The idea of Laplace tr. is to change an IVP into an algebra problem (no analysis needed after that!). We usually use t as the variable in the ODE.

Laplace transforms

The idea of Laplace tr. is to change an IVP into an algebra problem (no analysis needed after that!). We usually use t as the variable in the ODE.

The **Laplace transform** of a function $f(t)$ is a new function $F(s)$ determined by the definition below. Notice that the variable changed from t to s. People sometimes say that formulas with t are in the "time domain" and formulas with s are in the "frequency domain".

- Definition: $\mathscr{L}[f(t)] = \int_0^\infty f(t)e^{-st}\mathrm{d}t$
- In practice, we can just use a table of common Laplace transforms.

Laplace transforms

Time	Frequency
e^{at}	$\frac{1}{s-a}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
t^n	$\frac{n!}{s^{n+1}}$

We write, for example,

$$\mathscr{L}\left[e^{2t}\right] = \frac{1}{s-2},$$

and say "the Laplace transform of e^{2t} is $\frac{1}{s-2}$".

We can also go backwards: "the inverse Laplace transform of $\frac{24}{s^5}$ is t^4", written as

$$\mathscr{L}^{-1}\left[\frac{24}{s^5}\right] = t^4.$$

Laplace transforms

Time	Frequency
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t^n	$\dfrac{n!}{s^{n+1}}$

There are also some important properties of Laplace transforms:

- $\mathscr{L}\left[f(t)+g(t)\right] = F(s)+G(s).$
- $\mathscr{L}\left[c\cdot f(t)\right] = c\cdot F(s).$

Using the second bullet, the inverse Laplace transform of $\frac{8}{s^5}$ is

$$\mathscr{L}^{-1}\left[\frac{8}{s^5}\right] = \mathscr{L}^{-1}\left[\frac{1}{3}\cdot\frac{24}{s^5}\right] = \frac{1}{3}t^4.$$

Laplace transforms

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t^n	$\frac{n!}{s^{n+1}}$
$y'(t)$	$sY(s) - y(0)$

There are also some important properties of Laplace transforms:

- $\mathscr{L}\left[f(t) + g(t)\right] = F(s) + G(s)$.
- $\mathscr{L}\left[c \cdot f(t)\right] = c \cdot F(s)$.
- $\mathscr{L}\left[e^{at} f(t)\right] = F(s-a)$.
- $\mathscr{L}\left[t\, f(t)\right] = -F'(s)$.
- $\mathscr{L}\left[f'(t)\right] = s\,F(s) - f(0)$.

This last one uses both F and f. It is also why we can use $\mathscr{L}$ to solve ODEs!

Note: there is no nice formula for $\mathscr{L}\left[f(t)\, g(t)\right]$.

Example: Solve $y' + y = t$, $y(0) = 8$ using Laplace transf.

Time	Frequency
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$y'(t)$	$sY(s) - y(0)$

$$\mathscr{L}[y'] + \mathscr{L}[y] = \mathscr{L}[t]$$

$$sY - 8 + Y = \frac{1}{s^2}$$

$$Y = \frac{1+8s^2}{s^2+s^3} = \frac{9}{s+1} + \frac{-1}{s} + \frac{1}{s^2}$$ (partial fractions!)

$$y = 9\mathscr{L}^{-1}\left[\frac{1}{s+1}\right] - \mathscr{L}^{-1}\left[\frac{1}{s}\right] + \mathscr{L}^{-1}\left[\frac{1}{s^2}\right]$$

$$\boxed{y = 9e^{-t} - 1 + t}$$

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$$y' + a(t)\,y = f(t)$$

The solution to the similar ODE

$$y' + a(t)\,y = 0$$

is just $y = Ce^{-A(t)}$, where $A' = a$.

What if instead of a constant C we used $y = g(t)e^{-A(t)}$ for some $g(t)$?

Example: Solve $y' + y = t$ using variation of parameters.

Because $y' + y = 0$ has solution $y = C \cdot e^{-t}$, assume that $y(t) = g(t) \cdot e^{-t}$ for some function g.

The ODE is $y' + y = t$, so we need to use

$$(ge^{-t})' + ge^{-t} = t$$

Warm-up: (a) What is the derivative of $e^{2t}\sin(t)$?

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$$-e^{-t}g(t) + e^{-t}g'(t)$$

Example: Solve $y' + y = t$ using variation of parameters.

Because $y' + y = 0$ has solution $y = C \cdot e^{-t}$,
assume that $y(t) = g(t) \cdot e^{-t}$ for some function g.

The ODE is $y' + y = t$, so we need to use

$$(g \cdot (-e^{-t}) + g' \cdot e^{-t}) + ge^{-t} = t \quad \text{from warm-up}$$

...

...

$$g = te^{t} - e^{t} + C$$

Finally, $y = ge^{-t} = (te^{t} - e^{t} + C)e^{-t} = \boxed{t - 1 + Ce^{-t}}$

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$$y' + a(t)\,y = f(t)$$