Analysis 2

Tuesday, 21 May 2024

Warm-up: Re-write the ODE

$$ty' + t^3y + \sin(t) = 0$$

in the form $y' +$ ____ $y =$ ____.

Linear ODE

Last time

In standard form, a **first-order linear ODE** for $y(t)$ is

$$y' + a(t)\, y = f(t).$$

In standard form, a **second-order linear ODE** for $y(t)$ is

$$y'' + a(t)\, y' + b(t)\, y = f(t).$$

In standard form, a **third-order linear ODE** for $y(t)$ is

$$y''' + a(t)\, y'' + b(t)\, y' + c(t)\, y = f(t).$$

- We could write $y(t)$, $y'(t)$, etc., instead of just y, y'. Skipping some of the "(t)"s just makes the equation easier to read.
- We can use other letters: $x' + a(t)\, x = b(t)$ is a linear ODE for $x(t)$.

First-order linear

Last time

There are four tools we can (sometimes) use to solve first-order linear IVPs.

- 🤔 variation of parameters
- Laplace transforms
- integrating factor
- (coming soon)

$$y' + a(t)\,y = f(t)$$

The solution to $y' + a(t)\,y = 0$ is always $y = Ce^{-A(t)}$, where $A' = a$.

Assume $y = g(t)e^{-A(t)}$ and plug this into the ODE. Find g', then g, then y.

Example: $y' + y = t$. $a = 1 \to A = t$, so assume $y = g(t) \cdot e^{-t}$.

...

$g' = te^t \quad \to \quad g = te^t - e^t + C \quad \to \quad y = t - 1 + Ce^{-t}$

First-order linear

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$$y' + a(t)\,y = f(t)$$

Laplace transforms change an IVP into an algebra problem (no analysis needed after that!). We use a table of common $f(t) \longleftrightarrow F(s)$.

Example: $y' + y = t$, $y(0) = 8$ $\xrightarrow{\mathscr{L}}$ $(sY - 8) + Y = \frac{1}{s^2}$

↓ algebra

$y = 9e^{-t} - 1 + t$ $\xleftarrow{\mathscr{L}^{-1}}$ $Y = \frac{9}{s+1} + \frac{-1}{s} + \frac{1}{s^2}$

When (not) to use Laplace

Laplace transforms change, for example,

$$y' + 12y = \sin(5t), \;\; y(0) = 3$$

into

$$(sY - 3) + 12Y = \frac{5}{s^2 + 25}$$

very nicely. But some tasks don't work as well.

- An ODE without an initial condition needs $y(0) = C$, which makes partial fractions much harder.
- Initial condition with $t \neq 0$ will require a trick called "shifting".
- Non-constant coefficients: $\mathscr{L}\left[a(t) \cdot y(t)\right]$ is known only for some $a(t)$.

Time	Frequency
e^{kt}	$\frac{1}{s-k}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
t^n	$\frac{n!}{s^{n+1}}$
$f'(t)$	$sF(s) - f(0)$
$t \cdot f(t)$	$F'(s)$
$e^{at} \cdot f(t)$	$F(s-a)$

First-order linear

There are four tools we can (sometimes) use to solve first-order linear ODEs.

- variation of parameters
- Laplace transforms
- 🤔 integrating factor
- (coming soon)

$$y' + a(t)\,y = f(t)$$

The idea of an **integrating factor** is to multiply the entire ODE by some unknown function to make it nicer. We call this function $M(t)$—in some books, $\mu(t)$—and we will see what properties of $M(t)$ will be helpful to change the equation into one we can solve.

Example: Solve $y' + y = t$ using an integrating factor.

First step: $\mu(t) \cdot y'(t) + \mu(t) \cdot y(t) = \mu(t) \cdot t$

First-order linear

There are four tools we can (sometimes) use to solve first-order linear ODEs.

- variation of parameters
- Laplace transforms
- integrating factor
- 🤔 big formula

$$y' + a(t)\,y = f(t)$$

Big formula: The general solution to $y' + a(t)\,y = f(t)$ is always

$$y = \left(\int e^{A(t)} f(t)\,\mathrm{d}t\right) e^{-A(t)}$$

where $A'(t) = a(t)$.

Example: Solve $y' + y = t$ using the big formula.

$$y' + a(t)\,y = f(t)$$

$$\downarrow$$

$$y = \left(\int e^{A(t)} f(t)\,\mathrm{d}t\right) e^{-A(t)}$$

where $A'(t) = a(t)$

First-order linear

There are four tools we can (sometimes) use to solve first-order linear ODEs.

- variation of parameters
- Laplace transforms
- 🤔 integrating factor
- 🤔 big formula

$$y' + a(t)\,y = f(t)$$

Big formula: The general solution to $y' + a(t)\,y = f(t)$ is always

$$y = \frac{1}{M(t)} \int M(t) f(t)\,\mathrm{d}t$$

where $A'(t) = a(t)$ and $M(t) = e^{A(t)}$.

Task: Solve $x' + 10x = 5e^{3t}, x(0) = 2.$

We could use...

- Laplace transforms
- integrating factor
- variation of parameters
- big formula

All methods except Laplace will first give us the general solution x = ... + C... and then we'll use x(0) = 2 to get C.

Task: Solve $x' + 10x = 5e^{3t}$, $x(0) = 2$.

Time	Frequency
e^{kt}	$\frac{1}{s-k}$
$f'(t)$	$sF(s) - f(0)$

We could use...

- Laplace transforms

  $(sX - 2) + X = \mathcal{L}[5e^{3t}]$, solve that for X, then use partial fractions and the table to find $x = \mathcal{L}^{-1}[X]$.

- integrating factor

  $Mx' + 10Mx = 5Me^{3t}$

  Left side will be $Mx' + M'x = (Mx)'$ if $M' = 10M \to M = e^{10t}$.

- variation of parameters

  $x' + 10x = 0$ would give $x = Ce^{-10t}$, so assume $x = g(t)e^{-10t}$.

  ODE is now $(ge^{-10t})' + 10(ge^{-10t}) = 5e^{3t}$. → Product Rule!

- big formula

  To use this you have to memorize it.

Harder:

- $x' + 10x = 50t^2 + 7, \; x(0) = 2.$ Solution: $x = 5t^2 - t + \frac{4}{5} + \frac{6}{5}e^{-10t}$
  - This will need integration by parts for non-Laplace methods.
- $x' + 10x = 13\sin(2t), \; x(0) = 2.$ Soln: $x = \frac{5}{4}\sin(2t) - \frac{1}{4}\cos(2t) + \frac{9}{4}e^{-10t}$
  - This will need integration by parts OR a quadratic denominator (linear numerator) for partial fractions.
- $x' + \frac{1}{t}x = 5e^t, \; x(1) = 3.$ Solution: $x = 5e^t - \frac{5e^t}{t} + \frac{3}{t}$
  - Needs integration by parts. (Laplace is *very* difficult here.)

Laplace transforms

Last time

The **Laplace transform** of a function $f(t)$ is a new function $F(s)$ determined by the definition below. Notice that the variable changed from t to s. People sometimes say that formulas with t are in the "time domain" and formulas with s are in the "frequency domain".

- In practice, we can just use a table of common Laplace transforms, but where does this table come from?
- Definition: $\mathscr{L}\left[f(t)\right] = \lim_{M\to\infty}\int_0^M f(t)e^{-st}\mathrm{d}t$

Calculate the Laplace transforms of

$$g(t) = e^{3t}$$

using the definition $\mathscr{L}[f(t)] = \int_0^\infty f(t)e^{-st}\mathrm{d}t$.

$$G(s) = \int_0^\infty e^{3t}\, e^{-st}\, dt = \int_0^\infty e^{(3-s)t}\, dt$$

$$= \frac{1}{3-s}\, e^{(3-s)t}\Big|_0^\infty = 0 - \frac{1}{3-s}\, e^0 = \frac{1}{s-3}$$

if $s>3$ we can think of $e^{(3-s)\infty}$ as $e^{-\infty}$ or 0.
Technically this is $\lim_{N\to\infty} e^{(3-s)N}$.

Laplace transforms

Common functions	
e^{kt}	$\frac{1}{s-k}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
t^n	$\frac{n!}{s^{n+1}}$
1	$\frac{1}{s}$

Properties	
$f(t) + g(t)$	$F(s) + G(s)$
$c \cdot f(t)$	$c\ F(s)$
$t \cdot f(t)$	$F'(s)$
$e^{kt} \cdot f(t)$	$F(s-k)$
$f'(t)$	$s\ F(s) - f(0)$

These tables assume a, b, n are constants.

Summary of 1st order linear

For $y' + a(t)\,y = f(t)$, let $A(t)$ be any anti-derivative of $a(t)$.

- **Big formula:** $y = \left(\int e^{A(t)} f(t)\,\mathrm{d}t\right) e^{-A(t)}$.
- **Integrating factor:** Multiply the ODE by $M(t)$, then force the left-hand side to look like the Product Rule. (This will always lead to $M(t) = e^{A(t)}$.)
- **Variation of parameters:** Assume $y(t) = g(t)e^{-A(t)}$. Plug this into the ODE (using the Product Rule) and get formulas for g', then g, and finally y.
- **Laplace transforms:** Take the Laplace transform of the whole IVP. This will be an equation involving $Y = Y(s)$. Use basic algebra to solve for Y, then write Y as a sum of partial fractions, then use the inverse Laplace transform to get from $Y(s)$ back to $y(t)$.

Vocabulary

Partial differential equation or PDE

Ordinary differential equation or ODE

Initial value problem or IVP

Initial condition or IC or initial value

Order

Explicit solution

Implicit solution

Particular solution or specific solution

General solution

Direct

Autonomous

Separable

Linear

Homogeneous (linear)

Constant coefficients (linear)

Laplace transform

Fundamental solution set — we have not learned this yet

Characteristic equation — we have not learned this yet

Second-order Linear

In standard form, a first-order linear ODE for $y(t)$ is

$$a(t)\, y' + b(t)\, y = g(t).$$

In standard form, a **second-order linear ODE** for $y(t)$ is

$$a(t)\, y'' + b(t)\, y' + c(t)\, y = f(t).$$

coefficients

non-homogeneous term

Second-order Linear

In standard form, a first-order linear ODE for $y(t)$ is

$$a(t)\,y' + b(t)\,y = g(t).$$

In standard form, a **second-order linear ODE** for $y(t)$ is

$$a(t)\,y'' + b(t)\,y' + c(t)\,y = f(t).$$

- If $f(t) = 0$ (constant zero function), the ODE is **homogeneous**.
- If $f(t) \neq 0$, the ODE is **non-homogeneous**.
  - To solve a non-hom. ODE, we will first think about the hom. version.
- Very often we will look at ODEs with constant coefficients:

$$a\,y'' + b\,y' + c\,y = f(t)$$

Consider the homogeneous linear ODE

$$t^3y''' - 3t^2y'' + 4ty' - 4y = 0.$$

Given[1] that $y = t^4$ and $y = t$ are both solutions, …

- Is $y = 8t^4$ a solution?
- Is $y = t + t^4$ a solution?
- Is $y = t^7$ a solution?
- Is $y = 390\,t^7$ a solution?

1. You would *not* be expected to find this yourself. When we solve higher-order ODEs in this class, they will almost always have constant coefficients.