# Analysis 2

Tuesday, 4 June 2024

Warm-up: Describe all functions $y(t)$ for which $y'' = 40t^3$.

$$y'' = 40t^3$$

$$y' = 10t^4 + C$$

$$y = 2t^5 + Ct + D$$

# Higher-order

Usually an order $n$ differential equation will have $n$ arbitrary constants in its general solution.

This should be almost obvious for ODEs like

- Order 2: $y'' = -9.8 \quad \rightarrow \quad y = -4.9t^2 + C_1 t + C_2$
- Order 3: $y''' = 1 \quad \rightarrow \quad y = \frac{1}{6}t^3 + C_1 t^2 + C_2 t + C_3$

but it's also true for more complicated ones:

- Order 1: $y' = \dfrac{\cos(t)}{10y} \quad \rightarrow \quad y = \pm\sqrt{C + \frac{1}{5}\sin(t)}$
- Order 3: $t^3 y''' - 2\,ty' + 4y = 0 \quad \rightarrow \quad y = C_1 t^2 + \dfrac{C_2}{t^2} + C_3 t^2 \ln(t)$

# Higher-order Linear

In standard form, a first-order linear ODE for $x(t)$ is

$$a(t)\,x' + b(t)\,x = f(t).$$

In standard form, a **second-order linear ODE** for $x(t)$ is

$$a(t)\,x'' + b(t)\,x' + c(t)\,x = f(t).$$

coefficients — non-homogeneous term

Order $n$: $$\_\_x^{(n)} + \_\_x^{(n-1)} + \cdots + \_\_x' + \_\_x = 0.$$

# Higher-order Linear

In standard form, a first-order linear ODE for $x(t)$ is

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In standard form, a **second-order linear ODE** for $x(t)$ is

$$a(t)\,x'' + b(t)\,x' + c(t)\,x = f(t).$$

- If $f(t) = 0$ (constant zero function), the ODE is called **homogenous**.
- If not, the ODE is called **non-homogeneous**.
- Mostly we will look at ODEs with **constant coefficients**:

$$ax'' + bx' + cx = f(t).$$

Because $(f+g)' = f' + g'$ and $(cf)' = c \cdot f'$ for constant $c$,

> If $f(t)$ and $g(t)$ are solutions to a homogeneous linear ODE then any function
>
> $$\_\_f(t) + \_\_g(t),$$
>
> where the blanks are *numbers*, will also be a solution.

This is called the "principle of superposition".

- The third-order homogeneous linear ODE

$$y''' - \frac{3}{t}y'' - \frac{4}{t^2}y' - \frac{4}{t^3}y = 0,$$

has solutions $y = t$, $y = t\ln(t)$, and $y = t^4$ (don't worry about how I found those functions). The general solution to that ODE must therefore be

$$y = C_1 t + C_2 t\ln(t) + C_3 t^4.$$

Because $(f+g)' = f' + g'$ and $(cf)' = c \cdot f'$ for constant $c$,

If $f(t)$ and $g(t)$ are solutions to a homogeneous linear ODE then any function

$$\_\_f(t) + \_\_g(t),$$

where the blanks are *numbers*, will also be a solution.

This is called the "principle of superposition".

- For any order $n$ homogeneous linear ODE, there must be some functions $y_1(t), \ldots, y_n(t)$ such that the general solution to that ODE is
$$y = C_1 y_1(t) + C_2 y_2(t) + \cdots + C_n y_n(t).$$
- Those fns. form what is called a **fundamental set of solutions** for the ODE.
- So the real question is how to find these "fundamental solutions".
  - In this course, we will only answer this for constant coefficients.

Task 1a: For the ODE $x'' + 3x' - 10x = 0$, what functions of the form

$$x = e^{rt}$$

are solutions? Here $r$ is a constant.

$$(e^{rt})'' + 3(e^{rt})' - 10(e^{rt}) = 0$$

...

Answer: $x = e^{2t}$ and $x = e^{-5t}$

Task 1b: Give the general solution to $x'' + 3x' - 10x = 0$.

Answer: $x = C_1 e^{2t} + C_2 e^{-5t}$

# Characteristic ...

For a second-order homogeneous linear ODE with constant coefficients

$$ay'' + by' + cy = 0,$$

the **characteristic polynomial** is

$$ar^2 + br + c.$$

Because the letter $x$ might be used in the ODE, it is common to use a different letter (usually $r$) for the variable in this polynomial.

- For an ODE with order $n$, we get a degree $n$ polynomial.

Task 2: What functions have $y'' = -y$? (Char. polyn. $r^2 + 1$.)

Full list: any $C_1 \sin(t) + C_2 \cos(t)$.

Task 3a: Give the characteristic equation for the ODE $y'' - 6y' + 13y = 0$.

$r^2 - 6r + 13 = 0$

Task 3b: Solve the characteristic equation (that is, find the roots of the characteristic polynomial.)

$r = 3+2i, \quad r = 3-2i$

Task 3c: Solve the ODE. That is, describe all functions $y : \mathbb{R} \to \mathbb{R}$ for which $y'' - 6y' + 13y = 0$.

$x = C_1 e^{3t} \sin(2t) + C_2 e^{3t} \cos(2t)$

The general solution to an order $n$ homogeneous linear ODE is always

$$C_1 \underline{\qquad} + C_2 \underline{\qquad} + \cdots + C_n \underline{\qquad}.$$

- The issue is finding the functions to put in those blanks.
- If the ODE has constant coefficients, the roots of the characteristic polynomial will tell us what functions to use.

*Fact:* For each root $r = \lambda \pm \mu i$ with multiplicity $m$, the functions

- $e^{\lambda t}\sin(\mu t)$,
- $t\,e^{\lambda t}\sin(\mu t)$,
- $\vdots$
- $t^{m-1}\,e^{\lambda t}\sin(\mu t)$,
- $e^{\lambda t}\cos(\mu t)$,
- $t\,e^{\lambda t}\cos(\mu t)$,
- $\vdots$
- $t^{m-1}\,e^{\lambda t}\cos(\mu t)$,

are solutions, and this process gives a fundamental set of solutions to the ODE.

If $r = \lambda$ is a real number, we just have
$e^{\lambda t}\cos(0t) = e^{\lambda t}$ and $e^{\lambda t}\sin(0t) = 0$.

*Fact:* For a second-order linear homogeneous ODE with constant coefficients, there are only three possibilities for the general solution:

- Distinct real roots $\alpha, \beta$ → $C_1 e^{\alpha t} + C_2 e^{\beta t}$
- Complex roots $\lambda \pm \mu i$ → $C_1 e^{\lambda t} \sin(\mu t) + C_2 e^{\lambda t} \cos(\mu t)$
- Repeated real root $\lambda$ → $C_1 e^{\lambda t} + C_2 t e^{\lambda t}$

Example: Solve $x'' + 14x' + 49x = 0$.

$r^2 + 14r + 49 = 0$

$(r + 7)^2 = 0$

repeated root (mult. 2) → $\boxed{x(t) = C_1 e^{-7t} + C_2 t e^{-7t}}$

An order $2$ IVP will have *two* initial conditions. These could be two $y(\_\_) = \_\_$, but often one IC will use the derivative.

Example: Solve $\underbrace{y'' - 10y' + 29y = 0}_{\text{ODE}}$, $\underbrace{y(0) = 7}_{\text{IC}}$, $\underbrace{y'(0) = 30}_{\text{IC}}$.

Step 1. Find the general solution to $y''-10y'+29y=0$ using the characteristic polynomial.

Step 2. Find $C_1$ and $C_2$ using the initial conditions.

**or**

Step 1. Take Laplace transform of the entire ODE.

Step 2. Solve for $Y$ as sum of partial fractions.

Step 3. Get $y$ from $Y$.

# ODEs/IVPs we can solve

At this point, we have talked about how to solve…

- first-order separable (this includes direct and autonomous).
- first-order linear (regardless of whether it is homogeneous and/or has constant coefficients).
  - You can choose from VoP, IF, big formula, or (for IVP) Laplace.
- homogeneous linear with constant coefficients (any order, but $2$ is common).
  - Always use characteristic polynomial.

We have one more category to discuss:

- non-homogeneous linear with constant coefficients.