Analysis 2

Thursday, 6 June 2024
Monday, 10 June 2024

Warm-up: If $(Ae^{7t})' + Ae^{7t} = 3e^{7t}$, what is the number A?

$A = 3/8$

Last time

For any n^{th} order homogeneous linear ODE for $y(t)$, the general solution always looks like

$$y = C_1\_\_\_\_ + C_2\_\_\_\_ + \cdots + C_n\_\_\_\_.$$

- The only issue is finding the "fundamental solutions" to put in those blanks.
- If the ODE has constant coefficients, like

$$ay'' + by' + cy = 0,$$

then the roots of the characteristic polynomial

$$ar^2 + br + c = 0$$

will tell us what functions to use.

- Real root α → use $e^{\alpha t}$ for one of the blanks.
- Complex roots $\alpha \pm i\beta$ → use $e^{\alpha t}\sin(\beta t)$ and $e^{\alpha t}\cos(\beta t)$ for two blanks.
- If multiplicity $m > 1$, also multiply by t (then t^2, t^3,... up to t^{m-1}).

Example: Solve $x^{(5)} - 7x^{(4)} + 17x''' - 47x'' + 72x' + 144x = 0.$

This is the only hard part.

$$r^5 - 7r^4 + 17r^3 - 47r^2 + 72r + 144 = 0$$

$$(r+1)(r-4)(r-4)(r+3i)(r-3i) = 0$$

$$x = C_1e^{-t} + C_2e^{4t} + C_3te^{4t} + C_4\sin(3t) + C_5\cos(3t)$$

Order doesn't matter: $x = C_1\sin(3t) + C_2e^{-t} + \cdots$ is fine.

Also $x = C_1\sin(-3t) + \cdots$ is fine because C_1 can be $-C_1$.

- Real root α → use $e^{\alpha t}$ for one of the blanks.
- Complex roots $\alpha \pm i\beta$ → use $e^{\alpha t}\sin(\beta t)$ and $e^{\alpha t}\cos(\beta t)$ for two blanks.
- If multiplicity $m > 1$, also multiply by t (then t^2, t^3,... up to t^{m-1}).

Every homogeneous linear ODE with constant coefficients is solved exactly like this. Aside from factoring the polynomial, they can never be difficult tasks.

- In this class, we will mostly use 2^{nd} order ODEs (so 2^{nd} degree polynomial), and for 3^{rd} or higher the roots will be easy to find (look at ± divisors of the constant term in the polynomial).

There are two types of linear ODEs that are difficult:

- Non-constant coefficients — *too hard* for this course (unless first-order).
- Non-homogeneous — hard, but we will do it.

Homogeneous Linear

If we want to solve the IVP

$$y'' + y' - 20y = 0\,, \quad y(0) = 1, \quad y'(0) = 9$$

there are two options:

- use Laplace transforms; OR
- use characteristic polynomial to find general solution, then find C_1 and C_2.

If we want to solve the ODE

$$y'' + y' - 20y = 0$$

then Laplace transforms are not as good.

Non-homogeneous linear

If we want to solve the IVP

$$y'' + y' - 20y = 5t, \;\; y(0) = 1, \;\; y'(0) = 8$$

there are two options:

- use Laplace transforms;   OR
- use characteristic polynomial together with the "method of undetermined coefficients" (new) to find general solution, then find C_1 and C_2.

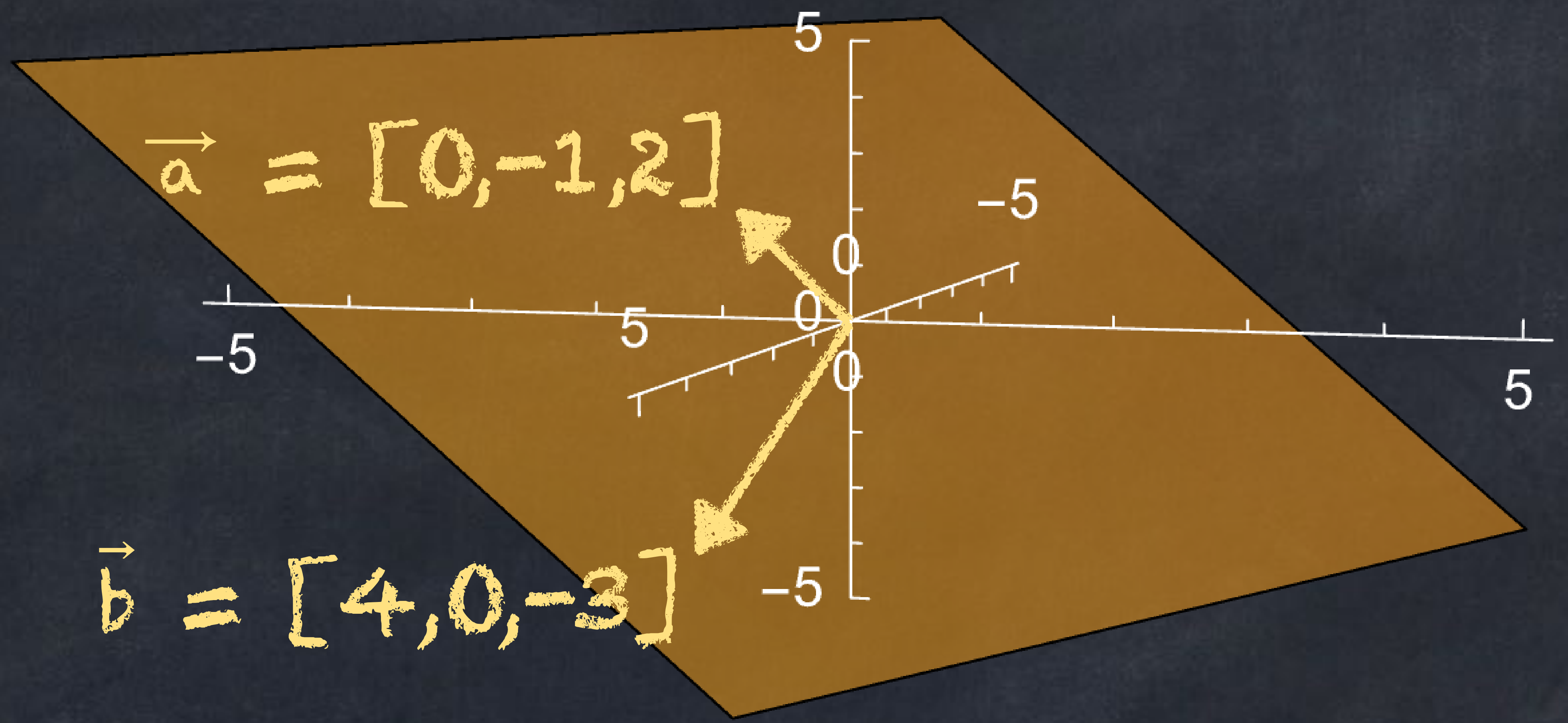
If we want to solve the ODE

$$y'' + y' - 20y = 5t$$

then Laplace transforms are not as good.

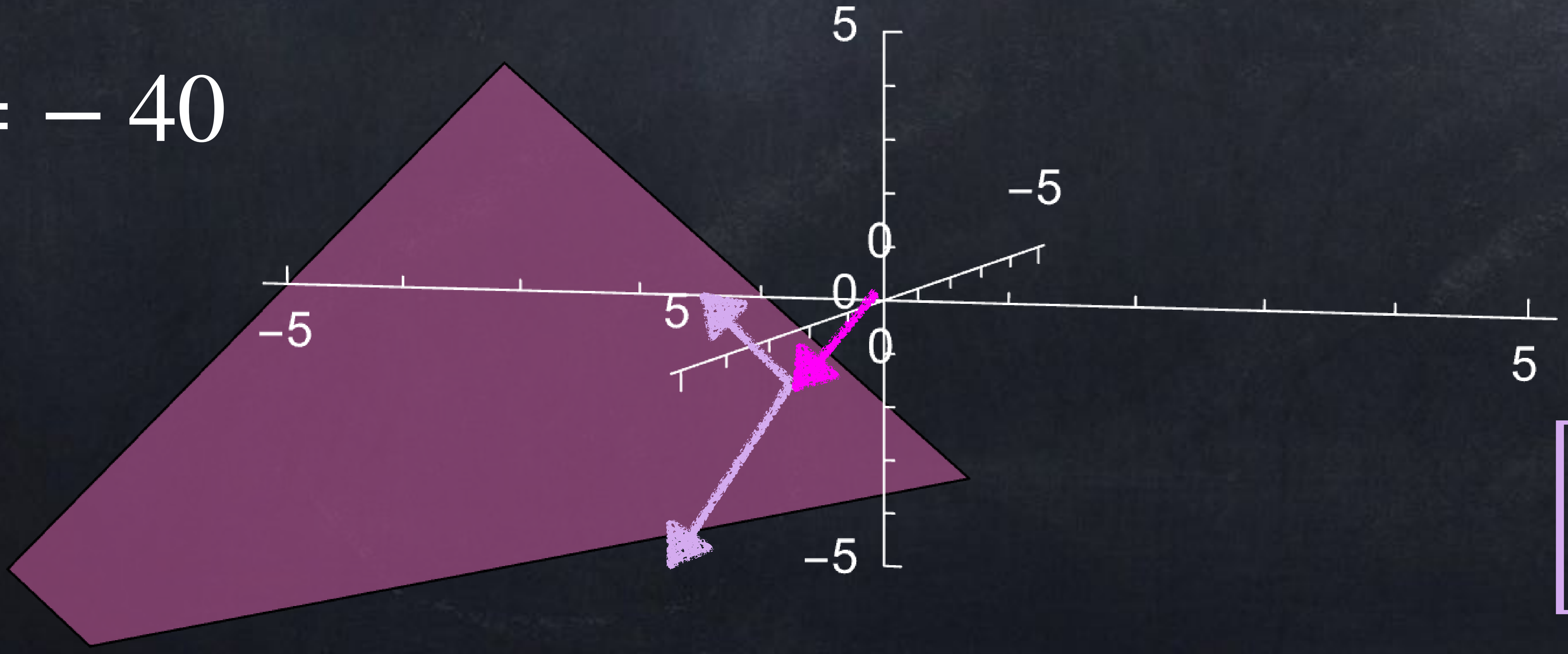
If this slide does not make sense to you, you can just ignore it 😎

$$3x + 8y + 4z = 0$$



$$\begin{bmatrix} x \\ y \end{bmatrix} = u\vec{a} + v\vec{b}$$

$$3x + 8y + 4z = -40$$



$$\begin{bmatrix} x \\ y \end{bmatrix} = \vec{P_0} + u\vec{a} + v\vec{b}$$

Non-homogeneous linear

Fact: If y_{NH} is one *particular* solution to a non-homogeneous linear ODE, then the *general* solution to that ODE is

$$y = y_{\text{NH}} + y_{\text{Hom}},$$

where y_{Hom} is the general solution to the corresponding homogeneous equation.

It may feel different but this is extremely similar to $\begin{bmatrix} x \\ y \end{bmatrix} = \vec{P_0} + u\vec{a} + v\vec{b}$.

There is a problem, though: in order to find the solution to the non-hom. equation, we need to already know one solution to the non-hom. equation!

- How can we find y_{NH} in the first place?

Non-homogeneous linear

To solve $ay'' + by' + cy = f(t)$, we guess the format of $y_{\mathrm{NH}}(t)$ from $f(t)$.

f	y_{NH}
$a\ e^{kt}$	$A\ e^{kt}$
$a \sin(kt)$	$A \sin(kt) + B \cos(kt)$
$a \cos(kt)$	$A \sin(kt) + B \cos(kt)$
$a\ t^k + \cdots$	$A\ t^k + \cdots + Y\,t + Z$

k is known from f(t).

A, B, ... are unknown.

* This assumes there is no "resonance" (next lecture).

Example 1: What is the format of y_{NH} for

$$y'' - 4y = 18e^{6t} \text{ ?}$$

Answer: $y_{\text{NH}} = Ae^{6t}$ for some number A.

Example 2: What is the format of y_{NH} for

$$y'' - 4y = 18e^{6t} + 10\sin(8t) \text{ ?}$$

Answer: $y_{\text{NH}} = A e^{6t} + B \sin(8t) + C \cos(8t)$
for some A, B, C.

In fact it will be
$y_{\text{NH}} = 9e^{6t} - \cos(8t) - \frac{1}{2}\sin(8t)$.

Example 1 again: Find one particular solution to

$$y'' - 4y = 18e^{6t}.$$

We know $y_{NH} = Ae^{6t}$ for some number A, but what is A?
Plug $y = Ae^{6t}$ into the ODE!

$$(Ae^{6t})'' - 4(Ae^{6t}) = 18e^{6t}$$

...

$$A = \frac{9}{16}$$

Answer: $y = \frac{9}{16}e^{6t}$ is one particular solution to the ODE.

Example 1 again: Find one particular solution to

$$y'' - 4y = 18e^{6t}.$$

$$y_{NH} = \frac{9}{16}e^{6t}$$

Example 1 again again: Find the general solution to

$$y'' - 4y = 18e^{6t}.$$

$$y'' - 4y = 0 \quad \rightarrow \quad r^2 - 4 = 0 \quad \rightarrow \quad r = \pm 2$$

so $y_{Hom} = C_1 e^{2t} + C_2 e^{-2t}$

$$\boxed{y = \frac{9}{16}e^{6t} + C_1 e^{2t} + C_2 e^{-2t}}$$

Hard task: Solve $y'' + 4y = t^2$, $y(0) = 1$, $y'(0) = 6$.

Option A:

Step 1. Find the general solution to $y'' + 4y = t^2$.

For non-hom., this requires (in either order)

1a. Find y_{Hom} for $y''+4y = 0$.

1b. Find y_{NH} for $y''+4y = t^2$.

Then $y = y_{Hom} + y_{NH}$.

Step 2. Find C_1, C_2.

Option B:

Laplace

From $f = y'$ we get $\mathscr{L}[y''] = s^2Y - sy(0) - y'(0)$.

t^n	$\dfrac{n!}{s^{n+1}}$
$f'(t)$	$s\,F(s) - f(0)$

Answer: $y = \frac{1}{4}t^2 - \frac{1}{8} + 3\sin(2t) + \frac{9}{8}\cos(2t)$