Analysis 2

Tuesday, 11 June 2024

Task (not fast): Find the partial fraction decomposition of $\dfrac{2s^2 - 13s + 36}{(s-5)^2(s+2)}$.

Last week

For any n^{th} order homogeneous linear ODE for $y(t)$, the general solution always looks like

$$y = C_1 \_\_\_\_ + C_2 \_\_\_\_ + \cdots + C_n \_\_\_\_ .$$

- The only issue is finding the "fundamental solutions" to put in those blanks.

- If the ODE has constant coefficients, like
$$ay'' + by' + cy = 0,$$
then the roots of the characteristic polynomial
$$ar^2 + br + c = 0$$
will tell us what functions to use.

Last week

Fact: For a second-order linear homogeneous ODE with constant coefficients, there are only three possibilities for the general solution:

- Distinct real roots α, β → $C_1 e^{\alpha t} + C_2 e^{\beta t}$
- Complex roots $\lambda \pm \mu i$ → $C_1 e^{\lambda t} \sin(\mu t) + C_2 e^{\lambda t} \cos(\mu t)$
- Repeated real root λ → $C_1 e^{\lambda t} + C_2 t e^{\lambda t}$

Example: Solve $x'' + 14x' + 49x = 0$.

$r^2 + 14r + 49 = 0$

$(r + 7)^2 = 0$

repeated root (mult. 2) → $x(t) = C_1 e^{-7t} + C_2 t e^{-7t}$

new

Laplace transforms	
e^{at}	$\frac{1}{s-a}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
t^n	$\frac{n!}{s^{n+1}}$

All these tables assume a, b, c, n are constants.

Laplace transforms	
$t\,e^{at}$	$\frac{1}{(s-a)^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$t\sin(bt)$	$\frac{2bs}{(s^2+b^2)^2}$
$t\cos(bt)$	$\frac{s^2+b^2}{(s^2+b^2)^2}$

Properties	
$f(t)+g(t)$	$F(s)+G(s)$
$c \cdot f(t)$	$c\,F(s)$
$t \cdot f(t)$	$-F'(s)$
$e^{at} \cdot f(t)$	$F(s-a)$
$f'(t)$	$s\,F(s)-f(0)$

Non-homogeneous linear

Extra Lecture

Fact: If y_{NH} is one *particular* solution to a non-homogeneous linear ODE, then the *general* solution to that ODE is

$$y = y_{\text{NH}} + y_{\text{Hom}},$$

where y_{Hom} is the general solution to the corresponding homogeneous equation.

This leaves a problem, though: in order to find the solution to the non-hom. equation, we need to already know a solution to the non-hom. equation!

How can we find y_{NH}? Answer: "Guess" based on the function in the ODE.

Non-homogeneous linear

To solve $ay'' + by' + cy = f(t)$, we guess the format of $y_{\text{NH}}(t)$ from $f(t)$.

f	y_{NH}
$a\ e^{kt}$	$A\ e^{kt}$
$a \sin(kt)$	$A \sin(kt) + B \cos(kt)$
$a \cos(kt)$	$A \sin(kt) + B \cos(kt)$
$a\ t^k + \cdots$	$A\ t^k + \cdots + Y\,t + Z$

k is known from f(t).
A, B, ... are unknown.

* This assumes there is no “resonance”.

Extra Lecture

Solve $x'' + 2x' - 3x = 5\cos(t)$, $x(0) = 1$, $x'(0) = 0$.

- Option 1: Find general soln. x (from last time: $x_{\text{Hom}} + x_{\text{NH}}$), then C_1 and C_2.
- Option 2: Laplace ($\mathscr{L}$), then lots of algebra, then $\mathscr{L}^{-1}$.

Char. polyn. $r^2 + 2r - 3$ equals 0 when $r = 1$ or $r = -3$,
so $x_{\text{Hom}} = C_1 e^{-3t} + C_2 e^{t}$.

From 5cost, we know $x_{\text{NH}} = A\cos(t) + B\sin(t)$ for some A,B.
$x''_{\text{NH}} + 2x'_{\text{NH}} - 3x_{\text{NH}} = 5\cos(t)$ $\rightarrow$ $A = -1$ and $B = \frac{1}{2}$,
so $x = C_1 e^{-3t} + C_2 e^{t} - \cos(t) + \frac{1}{2}\sin(t)$ solves the ODE.

$x(0) = 1$ and $x'(0) = 0$ $\rightarrow$ $C_1 = \frac{5}{8}$ and $C_2 = \frac{11}{8}$,
so $\boxed{x = \frac{5}{8}e^{-3t} + \frac{11}{8}e^{t} - \cos(t) + \frac{1}{2}\sin(t)}$ solves the IVP.

Solve $x'' + 2x' - 3x = 5\cos(t)$, $x(0) = 1$, $x'(0) = 0$.

- Option 1: Find general soln. x (from last time: $x_{\text{Hom}} + x_{\text{NH}}$), then C_1 and C_2.
- Option 2: Laplace ($\mathscr{L}$), then lots of algebra, then $\mathscr{L}^{-1}$.

Char. polyn. $r^2 + 2r - 3$ equals 0 when $r = 1$ or $r = -3$,
so $x_{\text{Hom}} = C_1 e^{-3t} + C_2 e^t$.

Solve system of 2 linear equations for A and B.

From $5\cos t$, we know $x_{\text{NH}} = A\cos(t) + B\sin(t)$ for some A,B.

$x''_{\text{NH}} + 2x'_{\text{NH}} - 3x_{\text{NH}} = 5\cos(t)$ ⟶ $A = -1$ and $B = \frac{1}{2}$,

so $x = C_1 e^{-3t} + C_2 e^t - \cos(t) + \frac{1}{2}\sin(t)$ solves the ODE.

Solve system of 2 linear eqn for C_1 and C_2.

$x(0) = 1$ and $x'(0) = 0$ ⟶ $C_1 = \frac{5}{8}$ and $C_2 = \frac{11}{8}$,

so $\boxed{x = \frac{5}{8}e^{-3t} + \frac{11}{8}e^t - \cos(t) + \frac{1}{2}\sin(t)}$ solves the IVP.

Solve $x'' + 2x' - 3x = 5\cos(t)$, $x(0) = 1$, $x'(0) = 0$.

- Option 1: Find $x = x_{\text{Hom}} + x_{\text{NH}}$, then C_1 and C_2.
- Option 2: Laplace ($\mathscr{L}$), then lots of algebra, then $\mathscr{L}^{-1}$.

Time	Frequency
e^{at}	1 / (s-a)
$\sin(at)$	$a / (s^2+a^2)$
$\cos(at)$	$s / (s^2+a^2)$
$f'(t)$	$s F(s) - f(0)$

Using $f = x$, we get $\mathscr{L}[x] = s X - x(0)$.

Using $f = x'$, we get $\mathscr{L}[x''] = s^2 X - s\, x(0) - x'(0)$.

$$(s^2X - 1s - 0) + 2(sX - 1) - 3X = \frac{5s}{s^2+1}$$

$$X = \frac{s^3 + 2s^2 + 6s + 2}{(s^2+1)(s^2+2s-3)} = \frac{5/8}{s+3} + \frac{11/8}{s-1} + \frac{(-1)s + 1/2}{s^2+1}$$

$$= \frac{5}{8}\frac{1}{s+3} + \frac{11}{8}\frac{1}{s-1} - \frac{s}{s^2+1} + \frac{1}{2}\frac{1}{s^2+1}$$

Solve system with 4 EQUATIONS and 4 UNKNOWNS for partial fractions.

so $x = \frac{5}{8}e^{-3t} + \frac{11}{8}e^{t} - \cos(t) + \frac{1}{2}\sin(t)$ solves the IVP.

Solve $2y'' - 6y' - 20y - 42e^{5t} = 0, \quad y(0) = 2, \; y'(0) = 3.$

- Option 1: Find general soln. $y = y_{\text{Hom}} + y_{\text{NH}}$, then C_1 and C_2.
- Option 2: Laplace ($\mathscr{L}$), then lots of algebra, then $\mathscr{L}^{-1}$.

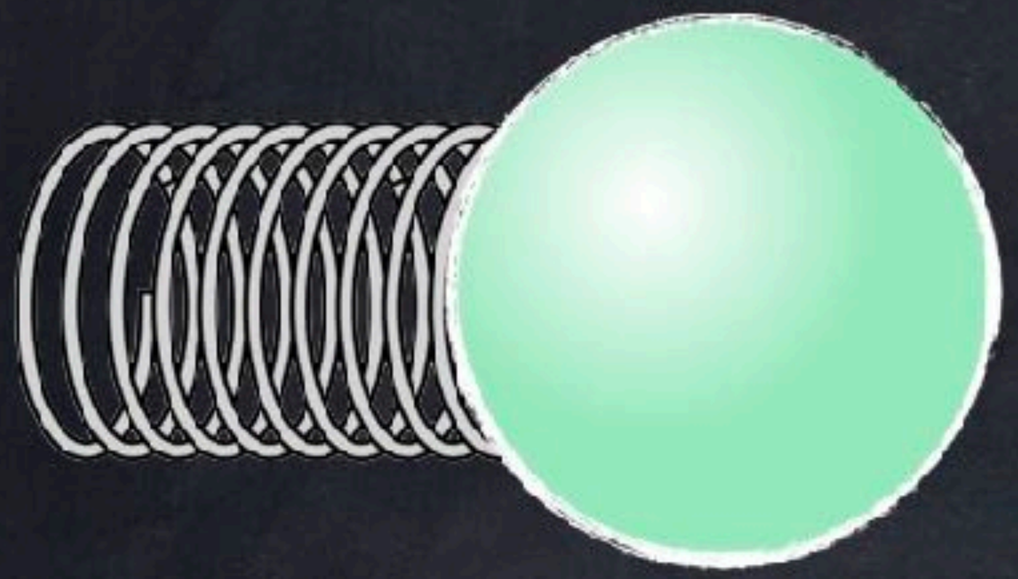
Before doing either method, it's good to re-write the ODE in standard form:

$$y'' - 3y' - 10y = 21e^{5t}$$

Now it's clear that this is non-homogeneous.

Answer: $\dfrac{10}{7}e^{-2t} + \dfrac{4}{7}e^{5t} + 3te^{5t}$

Some examples



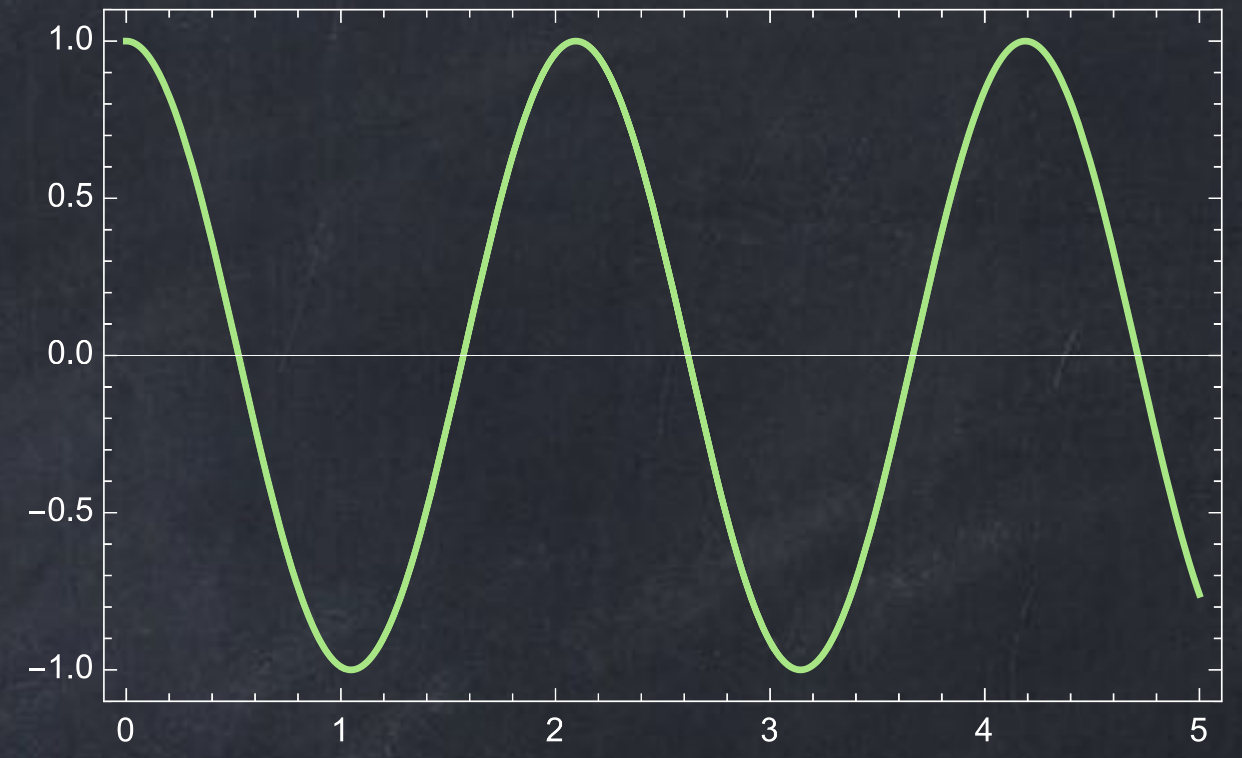
$$x'' + 9x = 0 \qquad x(0) = 1 \quad x'(0) = 0$$

$r^2 + 9 = 0$

$r = \pm 3i$

$x = C_1 \cos(3t) + C_2 \sin(3t)$

Undamped (pure) oscillator

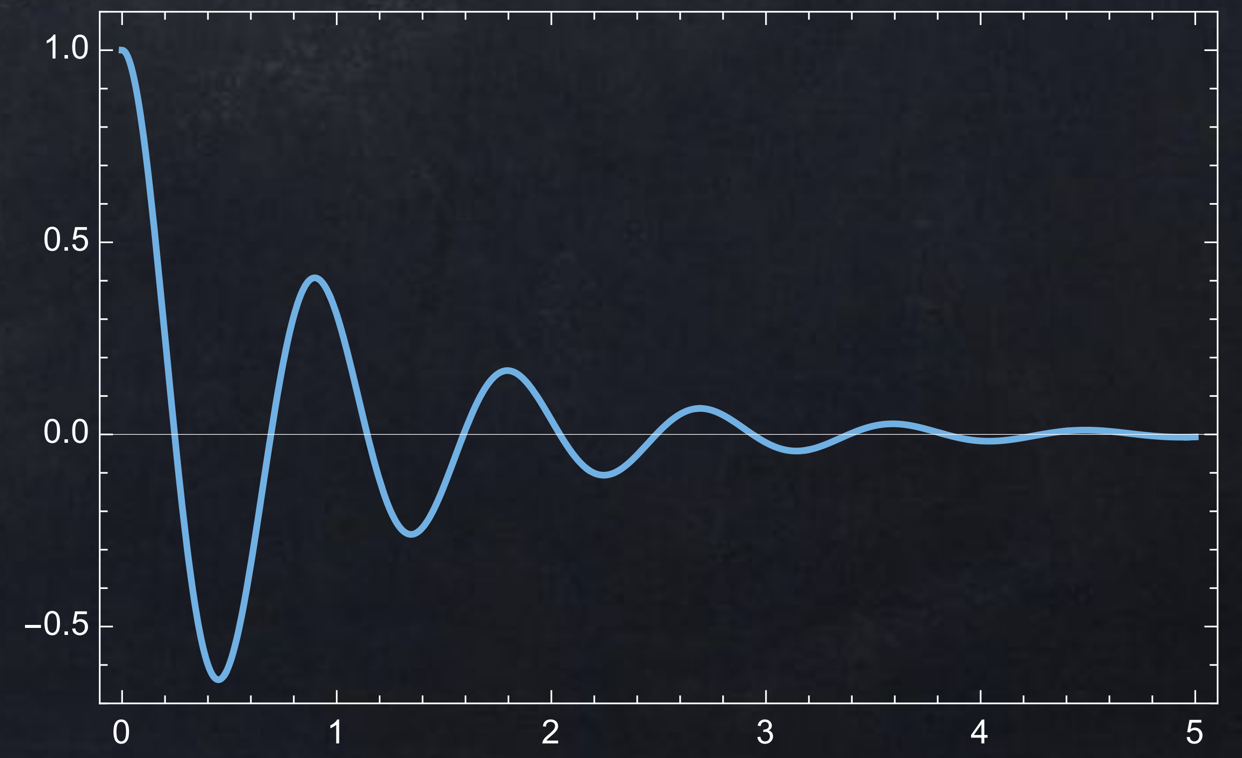


$$x'' + 2x' + 50x = 0$$

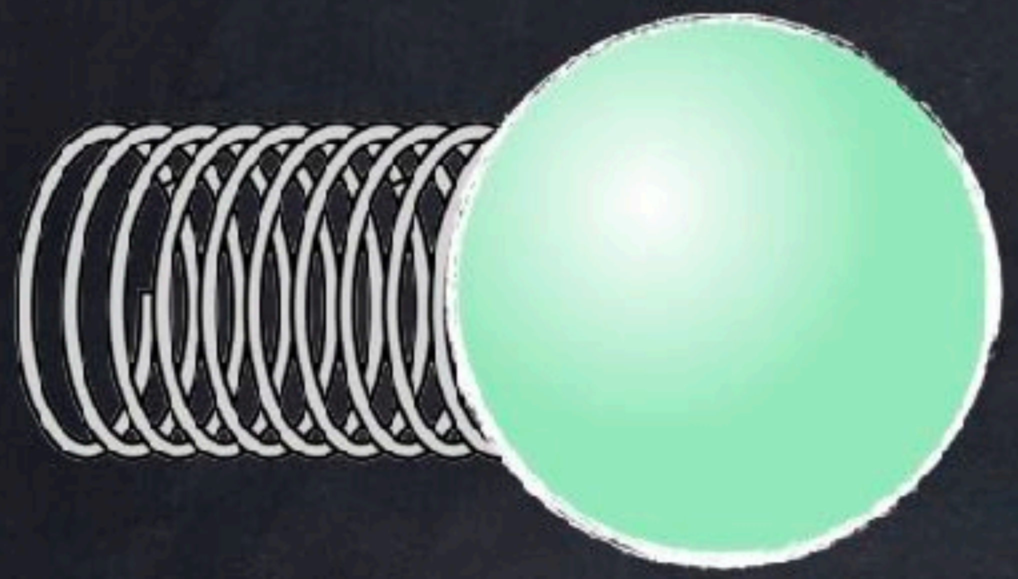
$r = -1 \pm 7i$

$x = e^{-t}\cos(7t) + \frac{1}{7}e^{-t}\sin(7t)$

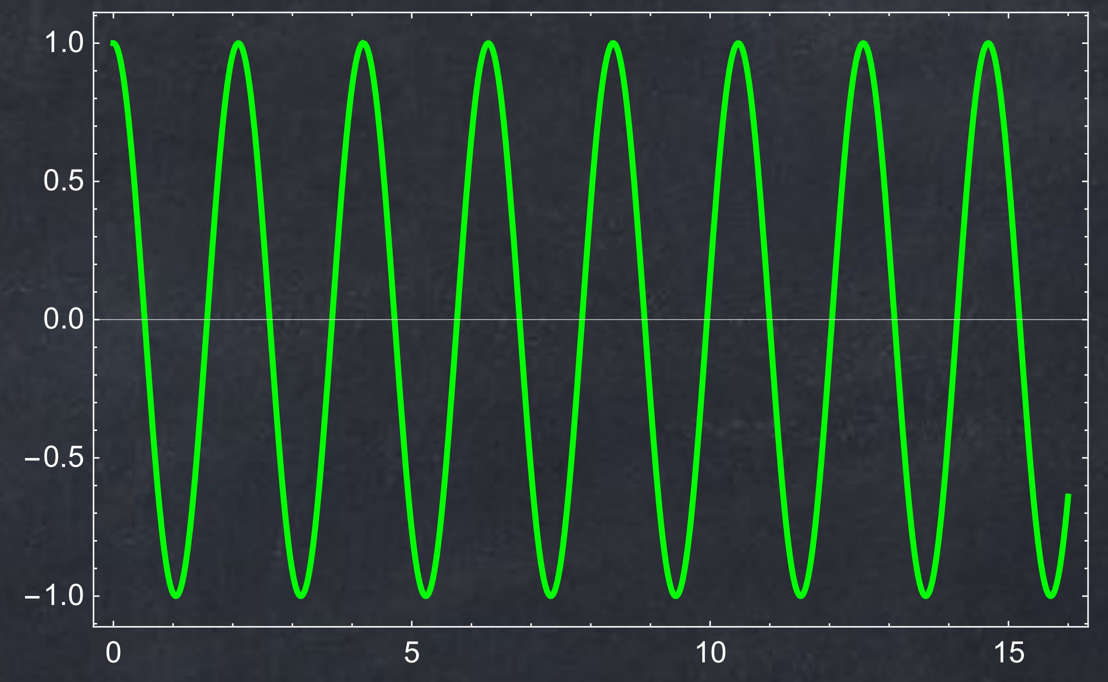
Damped oscillator



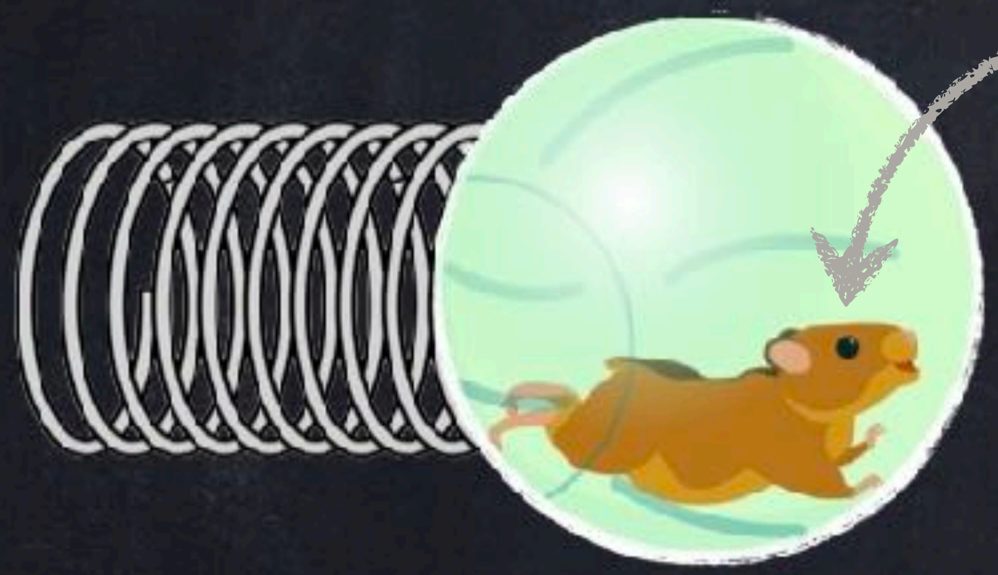
Some examples



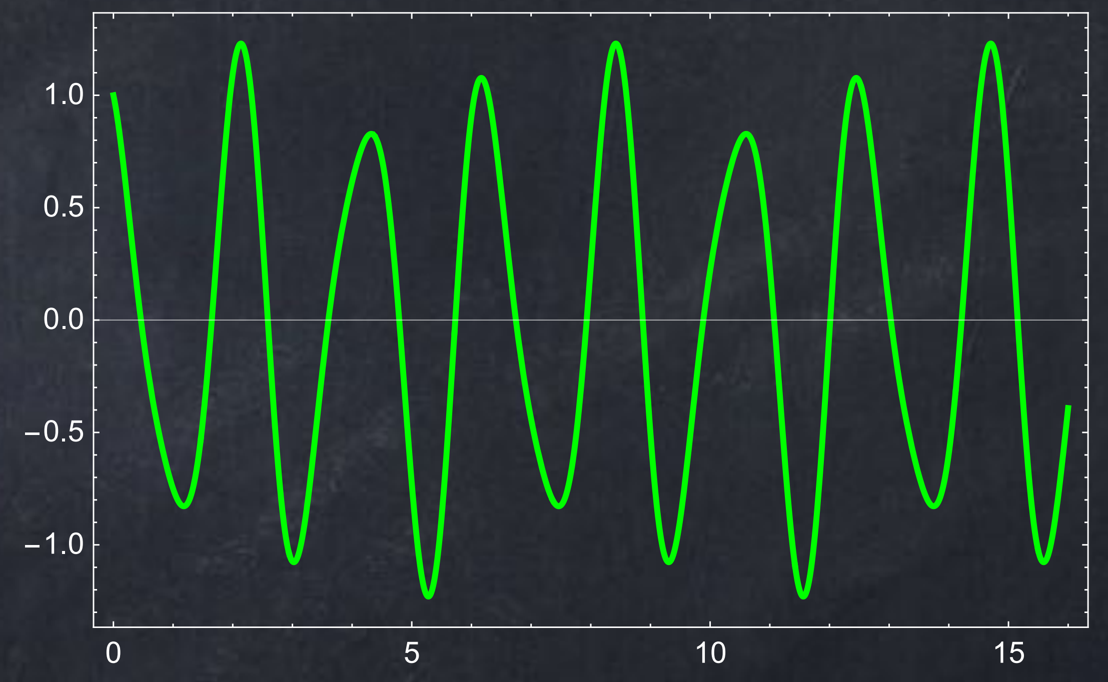
$$x'' + 9x = 0$$



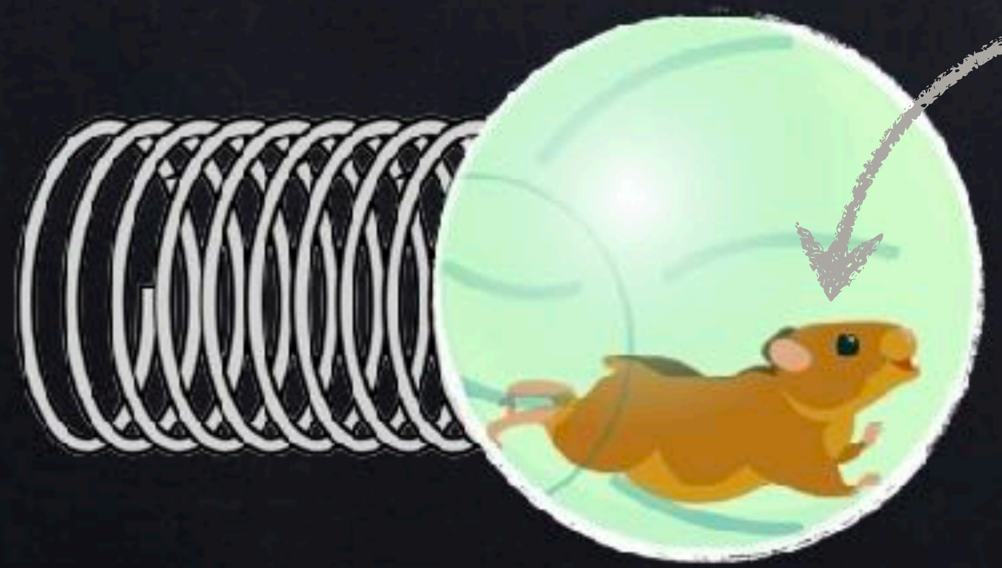
non-resonant hamster



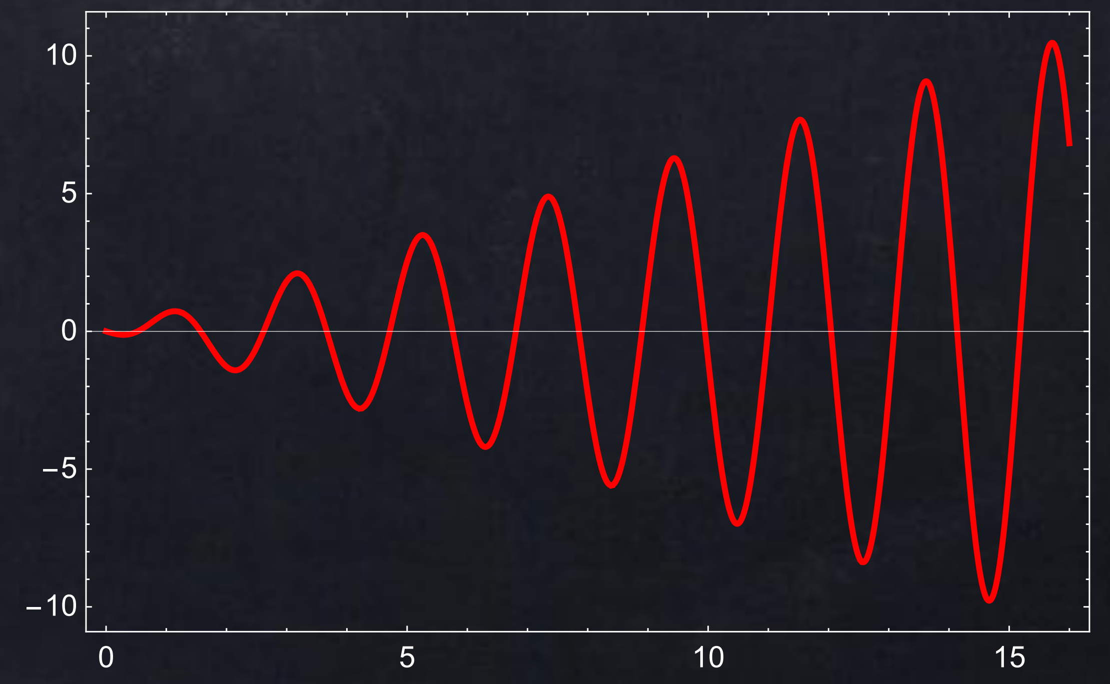
$$x'' + 9x = 4\sin(5t)$$



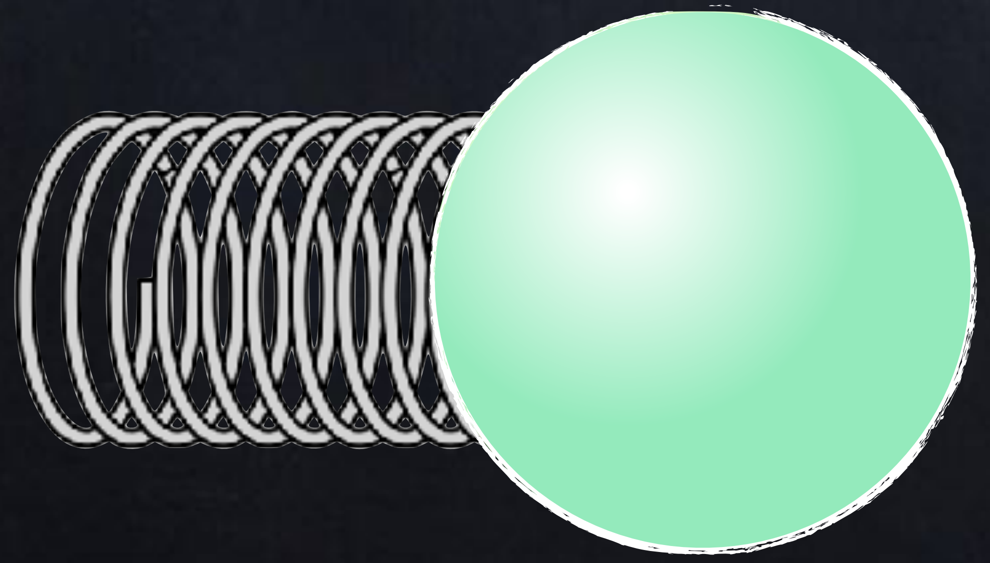
resonant hamster



$$x'' + 9x = 4\sin(3t)$$



A linear ODE has **resonance** if the frequency of the non-homogeneous term matches the frequency of the homogeneous solution.



When this happens, we must multiply the formula we would have used for y_{NH} by t.

Non-homogeneous linear

To solve $ay'' + by' + cy = f(t)$, we guess the format of $y_{\text{NH}}(t)$ from $f(t)$.

f	y_{NH} (no resonance)	y_{NH} with resonance
$a\, e^{kt}$	$A\, e^{kt}$	$A\, t\, e^{kt}$
$a \sin(kt)$	$A \sin(kt) + B \cos(kt)$	$A\, t \sin(kt) + B\, t \cos(kt)$
$a \cos(kt)$		
$a\, t^k + \cdots$	$A\, t^k + \cdots + Y\, t + Z$	(impossible)

If the characteristic polynomial for y_{Hom} has $(r-k)^2$ and $f = ae^{kt}$, then use $y_{\text{NH}} = A\, t^2\, e^{kt}$.

Systems of Linear ODEs

There is one final kind of differential equation for this class. Example:

$$\begin{cases} x' = \ \ 7x - 5y, \\ y' = 10x - 8y\,. \end{cases}$$

The solution to this is *two* functions, $x(t)$ and $y(t)$. Since this is an ODE, the functions will (both) have C_1 and C_2.

This kind of system can be solved

- by "eliminating x" to create a second-order ODE for $y(t)$, or
- by "eliminating y" to create a second-order ODE for $x(t)$, or
- by using eigenvalues and eigenvectors (but I will not show this method).

Example: Solve $\begin{cases} x' = 4x + 3y \\ y' = 2x + 9y \end{cases}$ by eliminating x.

Goal: $y'' + \_y' + \_y = 0$.

We can easily solve this for y.
Then the second eqn will give us x.

Example: Solve $\begin{cases} x' = 4x + 3y \\ y' = 2x + 9y \end{cases}$ by eliminating x.

$2x = y' - 9y$

$$y'' = 2x' + 9y'$$
$$y'' = 2(4x + 3y) + 9y'$$
$$y'' = 8x + 6y + 9y'$$
$$y'' = 4(2x) + 6y + 9y'$$
$$y'' = 4(y' - 9y) + 6y + 9y'$$

🙂 no x!

$$y'' = 4y' - 36y + 6y + 9y'$$
$$y'' = 13y' - 30y$$

Finally, move everything to one side:

$$y'' - 13y' + 30y = 0 \quad \rightarrow \quad y = C_1 e^{3t} + C_2 e^{10t}$$

Example: Solve $\begin{cases} x' = 4x + 3y \\ y' = 2x + 9y. \end{cases}$

- $\begin{cases} x = -3C_1e^{3t} + \frac{1}{2}C_2e^{10t} \\ y = C_1e^{3t} + C_2e^{10t} \end{cases}$
- $\begin{cases} x = \frac{1}{2}C_1e^{10t} - 3C_2e^{3t} \\ y = C_1e^{10t} + C_2e^{3t} \end{cases}$
- $\begin{cases} x = -3C_1e^{3t} + C_2e^{10t} \\ y = C_1e^{3t} + 2C_2e^{10t} \end{cases}$
- $\begin{cases} x = C_1e^{10t} - 3C_1e^{3t} \\ y = 2C_1e^{10t} + C_2e^{3t} \end{cases}$
- $\begin{cases} x = C_1e^{3t} + C_2e^{10t} \\ y = \frac{-1}{3}C_1e^{3t} + 2C_2e^{10t} \end{cases}$
- $\begin{cases} x = C_1e^{10t} + C_2e^{3t} \\ y = 2C_1e^{10t} - \frac{1}{3}C_2e^{3t} \end{cases}$

Each of these PAIRS is a correct answer to the task!

Time	Frequency
e^{at}	1 / (s-a)
$f'(t)$	$s\,F(s)$ - $f(0)$

Example 2: Solve $\begin{cases} x' = 7x - 5y, & x(0) = 9/2, \\ y' = 10x - 8y, & y(0) = 5. \end{cases}$

Since this is an IVP, we are looking for two specific functions.

- Options 1-2: Solve* ODE to get $x(t), y(t)$ with constants. Then find C_1, C_2.
- Option 3: Laplace.

* could be by eliminating x, or by eliminating y

Using $\mathscr{L}[x'] = sX - x(0)$, the Laplace transform of the entire first equation $x' = 7x - 5y$ is

$sX - 9/2 = 7X - 5Y.$

Th Laplace transform of $y' = 10x - 8y$ is

$sY - 5 = 10X - 8Y.$

Time	Frequency
e^{at}	1 / (s-a)
$f'(t)$	$s\,F(s) - f(0)$

Example: Solve $\begin{cases} x' = 7x - 5y, & x(0) = 9/2, \\ y' = 10x - 8y, & y(0) = 5. \end{cases}$

$$sX - 9/2 = 7X - 5Y \qquad (7-s)X - 5Y = -9/2$$
$$sY - 5 = 10X - 8Y \longrightarrow 10X - (s+8)Y = -5$$

Solve this system for X and Y. I used "Cramer's rule", but you can use other methods. Then do partial fraction decomposition.

$$X = \frac{\det\begin{pmatrix} -9/2 & -5 \\ -5 & s+8 \end{pmatrix}}{\det\begin{pmatrix} 7-s & -5 \\ 10 & -s-8 \end{pmatrix}} = \frac{9/2\,s + 11}{s^2 + s - 6} = \frac{1/2}{s+3} + \frac{4}{s-2}$$

$$Y = \frac{\det\begin{pmatrix} 7-s & -9/2 \\ 10 & -5 \end{pmatrix}}{\det\begin{pmatrix} 7-s & -5 \\ 10 & -s-8 \end{pmatrix}} = \frac{5s + 10}{s^2 + s - 6} = \frac{1}{s+3} + \frac{4}{s-2}$$

$\mathscr{L}^{-1}$

$$x = \tfrac{1}{2}e^{-3t} + 4e^{2t}$$
$$y = e^{-3t} + 4e^{2t}$$

ODE summary

first order separable ODE
- separation of variables

first order linear ODE
- direct formula

OR integrating factor

OR variation of param.

higher order linear ODE, constant coefficients, homogeneous
- characteristic polynomial

higher order linear ODE, cc., non-homogeneous
- find y_{Hom} (by char. polyn.), then find y_{NH} (by guess)

any linear ODE

★ if *given* a set of fundamental solutions

★ non-hom: if *given* y_{NH}

system of 2 first order cc. hom. linear ODEs
- order 2 by elimination

IVP summary

first order separable IVP
- separation of variables, then find C

higher order linear IVP, constant coefficients, homogeneous
- characteristic polynomial, then find C_1 and C_2

OR Laplace transforms

any IVP
★ if *given* the general solution

any linear IVP
★ if *given* a set of fundamental solutions
★ non-hom: if *given* y_{NH}

first order linear IVP
- direct formula, then C

OR integrating factor, C
OR variation of param., C
OR Laplace (only if cc.)

higher order linear IVP, cc., non-homogeneous
- find y_{Hom} (by char. polyn.), then find y_{NH} (by guess), then find C_1 and C_2

OR Laplace transforms

system of 2 first order cc. hom. linear IVPs
- order 2 by elim., then Cs

OR Laplace transforms