Analysis 2

19 March 2024

Warm-up: calculate the first partial derivatives of $f(x, y) = \frac{1}{4}y^4 + y\ln(x)$.

Derivatives

Last time

For a function with multiple inputs we can change x or change y (or both at once—more on that later), so we have multiple ways to take derivatives.

The **partial derivative of $f(x, y)$ with respect to x** can be written as any of

$$f'_x(x, y) \qquad f'_x \qquad D_x f \qquad \partial_x f \qquad \frac{\partial f}{\partial x}$$

and is what you get if you *think of every letter other than x as a constant*. Like with $f'(x)$ and $f'(a)$ from An. 1, we also have the **partial derivative of f with respect to x at the point (a, b)**, which is a single number; we write this as $f'_x(a, b)$.

There is also a partial derivative with respect to y (and to z if there are 3 inputs).

Derivatives

Last time

Example: for $f(x, y) = y^2 \sin(x)$ the partial derivatives are

- $f_x'(x, y) = y^2 \cos(x)$ and
- $f_y'(x, y) = 2y \sin(x)$.

These are similar to the Analysis 1 derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[17^2 \sin(x)\right] = 17^2 \cos(x) \text{ and } \frac{\mathrm{d}}{\mathrm{d}t}\left[t^2 \sin(4)\right] = 2t \sin(4).$$

There is also the **gradient**, which is the vector $\nabla f = (f_x')\hat{\imath} + (f_y')\hat{\jmath}$.

- For $f = y^2 \sin(x)$ we have $\nabla f(x, y) = \begin{bmatrix} y^2 \cos(x) \\ 2y \sin(x) \end{bmatrix}$.

We can also plug in specific points instead of (x, y) variables.

- $f_x'(3,0) = 9$
- $f_y'(3,0) = 0$
- $\nabla f(3,0) = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$.

Quiz 1 can have this and path integrals.

Last time

Suppose $f(x, y)$ describes the temperature at different positions.
If you stand at (a, b), you have the temperature $f(a, b)$.

- If you move east (right), your temperature changes at the rate $f_x'(a, b)$.
- If you move west (left), your temperature changes at rate $-f_x'(a, b)$.
- If you move north (up), your temperature changes at rate $f_y'(a, b)$.
- If you move south (down), your temperature changes at rate $-f_y'(a, b)$.

- What if you move northeast? Or south-southwest?

Linear approximation

Analysis 1: If you know that $f(7) = 13.8$ and $f'(7) = 0.5$,

- What is a good guess for $f(7.1)$? How confident are you of this guess?
- What is a good guess for $f(7.01)$? How confident are you of this guess?
- What is a good guess for $f(8)$? How confident are you of this guess?
- What is a good guess for $f(11)$? How confident are you of this guess?

Even though we actually want very small Δx, the value of f' is "the change in f *per unit change in* x" (meaning it's Δy for the case $\Delta x = 1$).

Directional derivative

The length of the vector $\vec{v} = [v_1, v_2]$ is $\vec{v} = \sqrt{v_1^2 + v_2^2}$.

A vector is is a **unit vector** if its length is 1.

Some people write a hat ˆ instead of an arrow → above unit vectors.

The **directional derivative of $f(x, y)$ at the point (a, b) in the direction of the unit vector $\vec{v} = [v_1, v_2]$** is a *number*

- written $f'_{\hat{u}}(a, b)$,
- spoken “the derivative of F in the direction U”,
- equal to $u_1 f'_x(a, b) + u_2 f'_y(a, b)$, but it is better to write this using vectors!

Directional derivative

The **directional derivative** of f in the direction of the **unit** vector $\hat{u}$ is

$$f'_{\hat{u}} = \hat{u} \cdot \nabla f$$

This formula only works when $\hat{u}$ is a *unit* vector (length 1)!

- For the direction of any vector $\vec{v}$, use $\hat{u} = \dfrac{\vec{v}}{\vec{v}}$.

Work

eportal.pwr.edu.pl/cours

SAMPLE QUIZ

You can use this to see how ePortal's online tasks work. This does not affect grades at all. This quiz contains several tasks (the real extra point quizzes will be one or two tasks). You can attempt this quiz multiple times (but the real quizzes only once).

Quiz points

Quiz 1 extra point

Quiz 1 topics: path integrals; *calculating* partial derivatives and gradients; any previous topics

Quiz 2 extra point

Quiz 2 topics: applying partial derivatives and gradients to solve tasks (e.g., local min and max); any previous topics

Quiz 3 extra point

Quiz 3 topics: double-integrals; any previous topics

Calendar

1510 List 0.pdf – Page 1 of 3

Analysis 2, Summer 2024

List 0

"Previous topics"

Basic algebra

1. Find the two values of r for which $r^2 - 2r - 15 = 0$.
2. Find the two (complex) values of r for which $r^2 - 2r + 15 = 0$.
3. (a) Solve $\sqrt{y} = \sin(x)$ for y. (b) Solve $\ln(y) = \sin(x)$ for y.
4. Solve $\ln(x) - 3 = 7t$ for x.

(e) is it possible that $\vec{u} \cdot \vec{v} = 4.61$?

(f) is it possible that $\vec{u} \cdot \vec{v} = -\sqrt{17}$?

(g) is it possible that $\vec{u} \cdot \vec{v} = -\sqrt{7}$?

20. If $|\vec{u}| = 1$ and $|\vec{w}| = 7$, describe ALL possible values that $\vec{u} \cdot \vec{w}$ could have.
21. If $|\vec{u}| = 1$ and $\vec{n} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$,
   (a) what is the largest possible value that $\vec{u} \cdot \vec{n}$ could have?
   (b) give an example of a vector $\vec{u}$ such that $\vec{u} \cdot \vec{n}$ has the value from part (a).
   (c) give an example of a vector $\vec{u}$ such that $\vec{u} \cdot \vec{n} = 0$.
22. Write $\dfrac{5x+6}{x^2-6x+8} = \dfrac{5x+6}{(x-2)(x-4)}$ as a sum of partial fractions.
   That is, find A and B such that
   $$\frac{5x+6}{x^2-6x+8} = \frac{A}{x-2} + \frac{B}{x-4}.$$
23. Write $\dfrac{2x-6}{(x-2)(x+6)}$ as a sum of partial fractions.
24. Write $\dfrac{36}{x^3+9x^2+18x}$ as a sum of partial fractions.

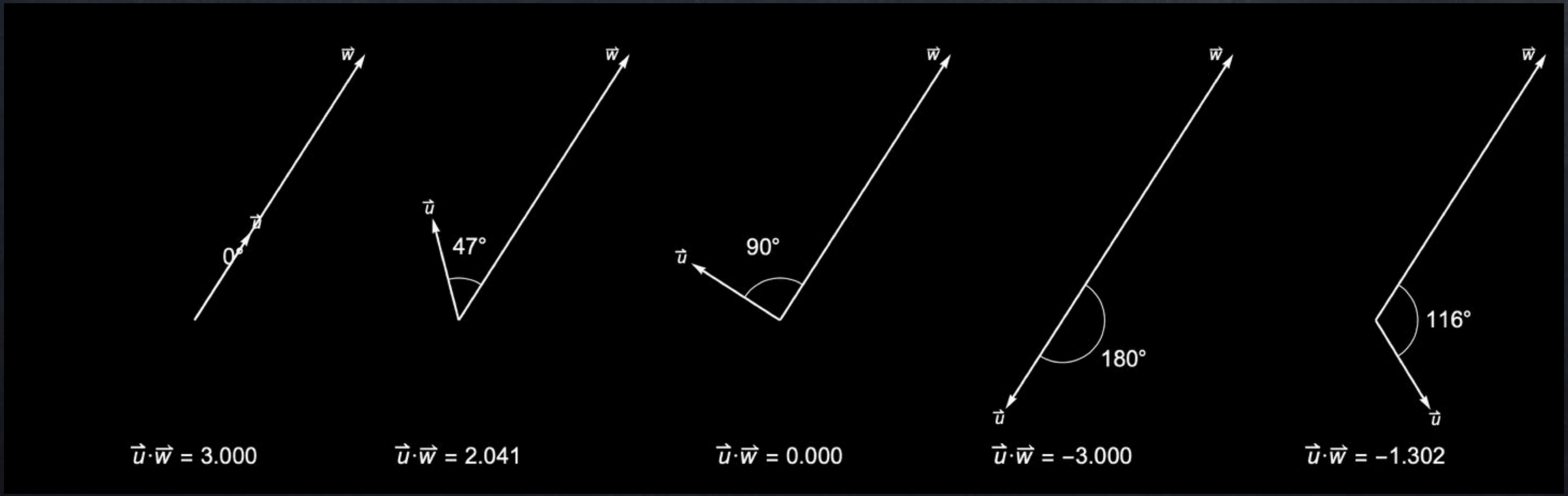
Dot product

If $\vec{u}$ has length 1 and $\vec{w}$ has length 3, what can $\vec{u} \cdot \vec{w}$ possibly be?

Answer: any value between −3 and 3.

Why? $\vec{u} \cdot \vec{w} = 3\cos(\theta)$.

$$\vec{u} \cdot \vec{w} = \vec{u} \quad \vec{w} \; \cos(\theta)$$



$$\nabla f = [f'_x, f'_y]$$

$$f'_{\hat{u}} = \hat{u} \cdot \nabla f$$

Task 1: Calculate the derivative of $x^2 + 3y^2$ at the point $(4,1)$ in the direction of $[-9, 9]$.

Task 2: What is the largest possible value of $f'_{\hat{u}}(4,1)$ for $f(x, y) = x^2 + 3y^2$ if you can choose any direction $\hat{u}$?

Task 3: In what direction is $f'_{\hat{u}}(4,1)$ the largest for $f(x, y) = x^2 + 3y^2$? (This is the "direction of steepest ascent".)

Derivatives

For $f(x, y)$ we have

- partial derivative with respect to x,
- partial derivative with respect to y,
- gradient,
- directional derivative.

These can be computed at a single point or thought of as a new function of x and y (but for $f'_{\hat{u}}$ we will only look at one point at a time).

What can we use these derivatives for?

Critical points

Analysis 1:

- A critical point (or CP) of $f(x)$ is an x-value where f' is zero or undefined.
- Critical points are often—but not always—locations of local extrema.
- Min or max? The *First Derivative Test* uses the sign of f' to the left and right of a CP. The *Second Derivative Test* uses the sign of f'' at the CP exactly.

Analysis 2:

- A **critical point** of $f(x, y)$ is a point (x, y) where ∇f is zero or undefined.
    - Note $\vec{0}$ is $0\hat{\imath} + 0\hat{\jmath}$, so "$\nabla f = \vec{0}$" means $f_x(x, y) = 0$ AND $f_y(x, y) = 0$.
    - We have to solve a system of equations to find the CP of $f(x, y)$!
- The *Second Derivative Test* uses ? ….

Critical points

The definition "where $\nabla f = \vec{0}$ or is undefined" for critical point works not only for $f(x, y)$ but also for $f(x, y, z)$, in which case

the gradient vector is $\nabla f = \begin{pmatrix} f'_x \\ f'_y \\ f'_z \end{pmatrix}$ and the zero vector is $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

We can do the same for $f(x_1, x_2, \ldots, x_{100})$, but of course I will never ask you to deal with such functions by hand.

Example: Find the critical points of $f(x, y) = x^3 + x^2y - y$.

Second derivatives

There are at least four kinds of “first derivatives” for $f(x, y)$:

- partial derivative with respect to x,
- partial derivative with respect to y,
- gradient,
- directional derivative.

There are several kinds of second derivatives for $f(x, y)$ also.

Second derivatives

There are several kinds of second derivatives for $f(x, y)$:

- second partial derivative with respect to x

$$f''_{xx} = (f'_x)'_x = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial x^2},$$

- second partial derivative with respect to y

$$f''_{yy} = (f'_y)'_y = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial^2 f}{\partial y^2},$$

- mixed partial derivatives,
- Hessian.

Second derivatives

There are several kinds of second derivatives for $f(x, y)$:

- second partial derivative with respect to x
- second partial derivative with respect to y
- mixed partial derivatives,

$$f''_{xy} = (f'_x)'_y = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial y \partial x}$$

and

$$f''_{yx} = (f'_y)'_x = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial^2 f}{\partial x \partial y},$$

- Hessian.

Second derivatives

There are several kinds of second derivatives for $f(x, y)$:

- second partial derivative with respect to x
- second partial derivative with respect to y
- mixed partial derivatives
- Hessian

$$\mathbf{H}f = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix}.$$

This is a <u>matrix</u> (similar to how ∇f is a vector).

We only actually have to calculate 3 of the 4 entries because the mixed partial derivatives are equal.

Example: Calculate all four second derivatives for $f(x, y) = ye^{2x+8y}$.

- $f''_{xx} = 4ye^{2x+8y}$
- $f''_{yy} = (64y+9)e^{2x+8y}$
- $f''_{xy} = (16y+2)e^{2x+8y}$
- $f''_{yx} = (16y+2)e^{2x+8y}$

Symmetry of second derivatives

If the second derivatives of $f(x, y)$ are continuous, then

$$f''_{xy} = f''_{yx}.$$

Example: Calculate $\nabla f(-4, 1)$ and $\mathbf{H}f(-4, 1)$ for $f(x, y) = ye^{2x+8y}$.

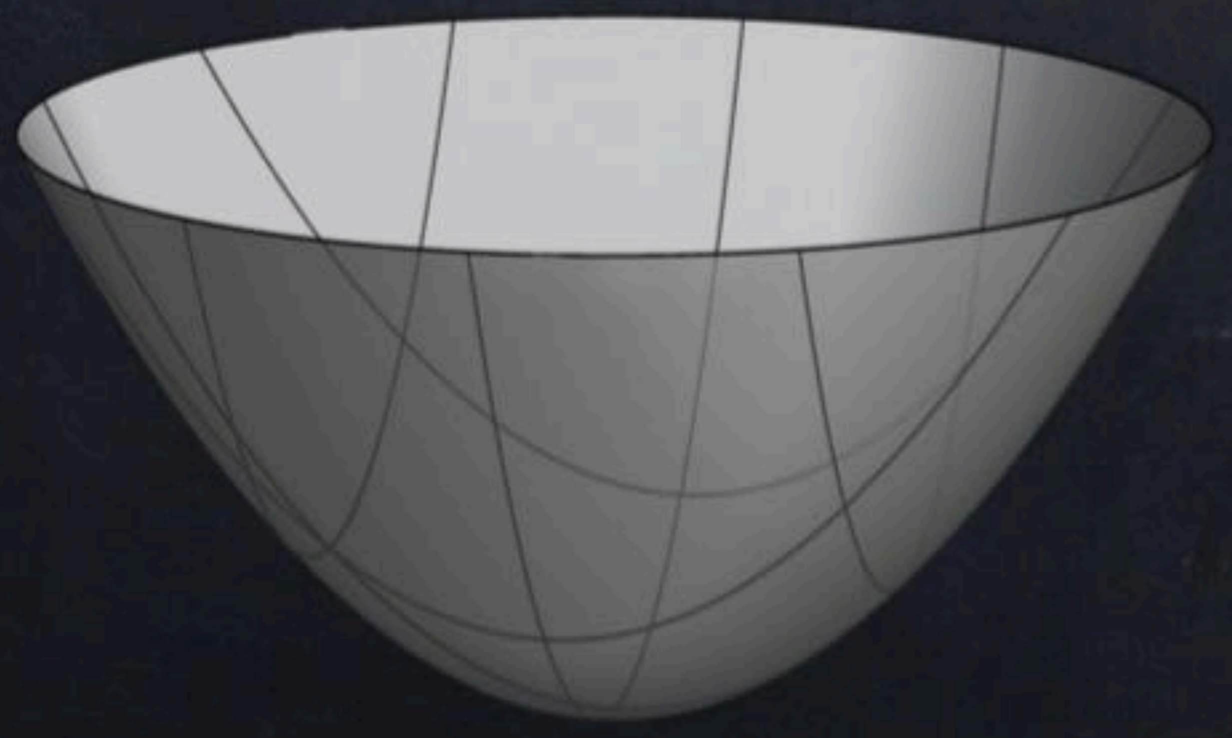
Second Derivative Test

1. To find the critical points of $f(x, y)$: solve $\nabla f = \vec{0}$ or undefined.

v1

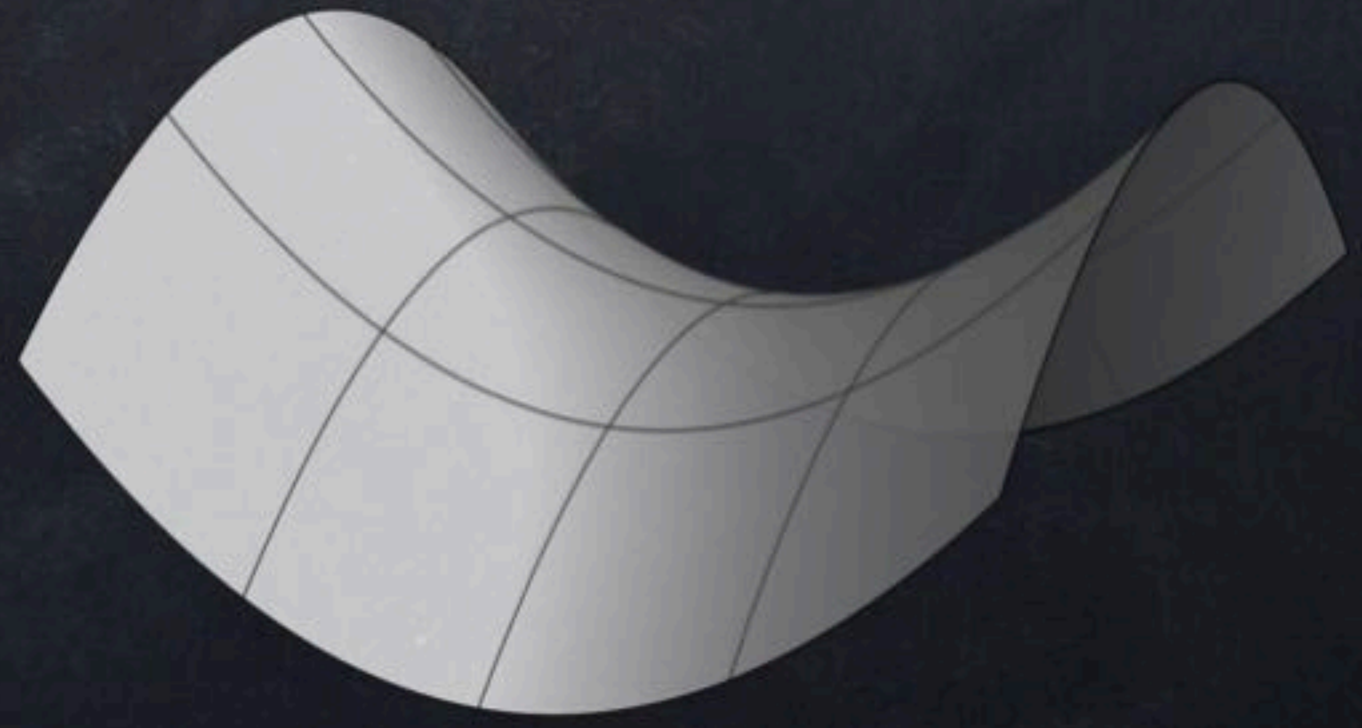
To classify the critical points:

2. Compute $\mathbf{H}f = \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{bmatrix}$ at each CP and let λ_1, λ_2 be its eigenvalues.
3. If $\lambda_1, \lambda_2 > 0$ then the CP is a LOCAL MIN.
   If $\lambda_1, \lambda_2 < 0$ then the CP is a LOCAL MAX.
   If λ_1, λ_2 have different ± signs then the CP is a SADDLE.



local min

local max



saddle

Second Derivative Test

1. To find the critical points of $f(x, y)$: solve $\nabla f = \vec{0}$ or undefined.

To classify the critical points:

v2

2. Compute $\mathbf{H}f = \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{bmatrix}$ at each CP.
3. If $\det(\mathbf{H}f) > 0$ and $\boxed{f''_{xx} > 0}$ then the CP is a LOCAL MIN.
   If $\det(\mathbf{H}f) > 0$ and $\boxed{f''_{xx} < 0}$ then the CP is a LOCAL MAX.
   If $\det(\mathbf{H}f) < 0$ then the CP is a SADDLE.

You can check f''_{yy} instead (it will have the same sign as f''_{xx} if det>0).

If $\det(\mathbf{H}f) = 0$ then the test doesn't tell us what kind of CP this is.

Second Derivative Test

1. To find the critical points of $f(x, y)$: solve $\nabla f = \vec{0}$ or undefined.

To classify the critical points:

v3

2. Compute $f''_{xx}, f''_{xy}, f''_{yx}, f''_{yy}$ at each CP.
3. If $f''_{xx}f''_{yy} - (f''_{xy})^2 > 0$ and $f''_{xx} > 0$ then the CP is a LOCAL MIN.
   If $f''_{xx}f''_{yy} - (f''_{xy})^2 > 0$ and $f''_{xx} < 0$ then the CP is a LOCAL MAX.
   If $f''_{xx}f''_{yy} - (f''_{xy})^2 < 0$ then the CP is a SADDLE.

You can check f''_{yy} instead (it will have the same sign as f''_{xx} if det>0).

If $f''_{xx}f''_{yy} - (f''_{xy})^2 = 0$ then the test doesn't tell us what kind of CP this is.

Example 1: Find and classify the critical points of $x^3 - 3xy + 8y^3$.

Task 2: Find and classify the critical points of $x^2 + 8y^2 - xy^2$.

CP	D	f''_{xx}	type