Analysis 2

9 April 2024

Warm-up: $\int_2^5 kx^2 \, \mathrm{d}x.$

Iterated Integrals

Last week

An **iterated integral** requires evaluating one integral after another. They will always be definite integrals.

- The "inside" integral can give a formula as its answer.
- The "outside" integral will usually give a number as the answer.

Example from last week $\int_0^8 \int_0^1 3xe^{xy}\,\mathrm{d}y\,\mathrm{d}x = 3e^8 - 27$ because

- Inside: $\int_0^1 3xe^{xy}\,\mathrm{d}y = 3e^{xy}\Big|_{y=0}^{y=1} = 3e^{x\cdot 1} - 3e^{x\cdot 0} = 3e^x - 3.$
- Outside: $\int_0^8 (3e^x - 3)\,\mathrm{d}x = 3e^x - 3x\Big|_{x=0}^{x=8} = 3e^8 - 27.$

We can calculate $\int_1^3 \int_2^5 yx^2 \, dx \, dy = \int_1^3 (39y) dy = \boxed{156}$

First, $\int_2^5 yx^2 \, dx = 39y$, just like the warm-up $\int_2^5 kx^2 \, dx = 39k$.

Then $\int_1^3 39y \, dy = 156$, just like $\int_1^3 39t \, dt = 156$ (or $\int_1^3 39x \, dx$).

We can calculate $\int_1^3 \int_2^5 yx^2 \, dx \, dy = \int_1^3 (39y) dy = 156.$

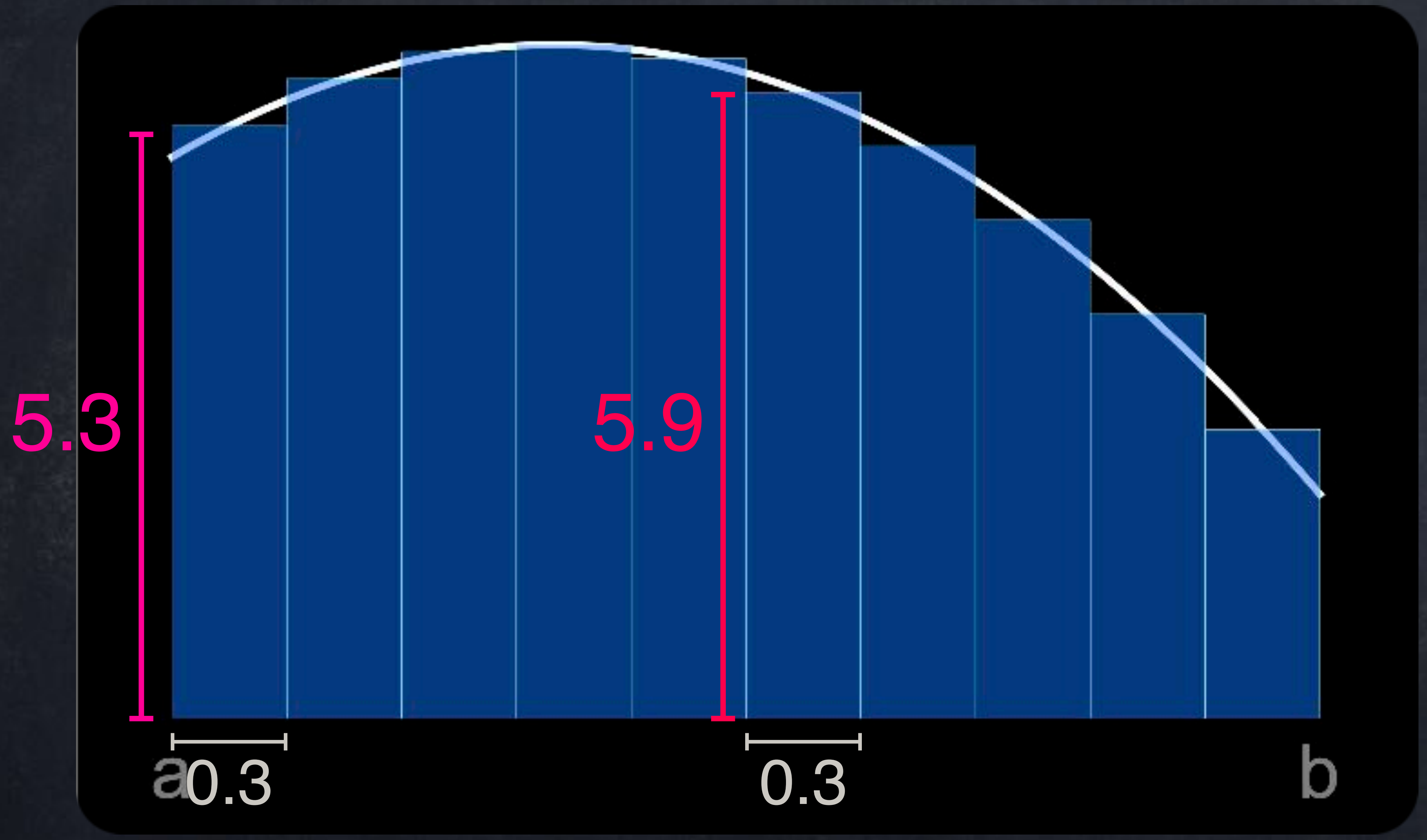
Front students: $\int_1^3 \int_2^5 yx^2 \, dy \, dx = \int_1^3 (\frac{21}{2}x^2) dx = 91.$

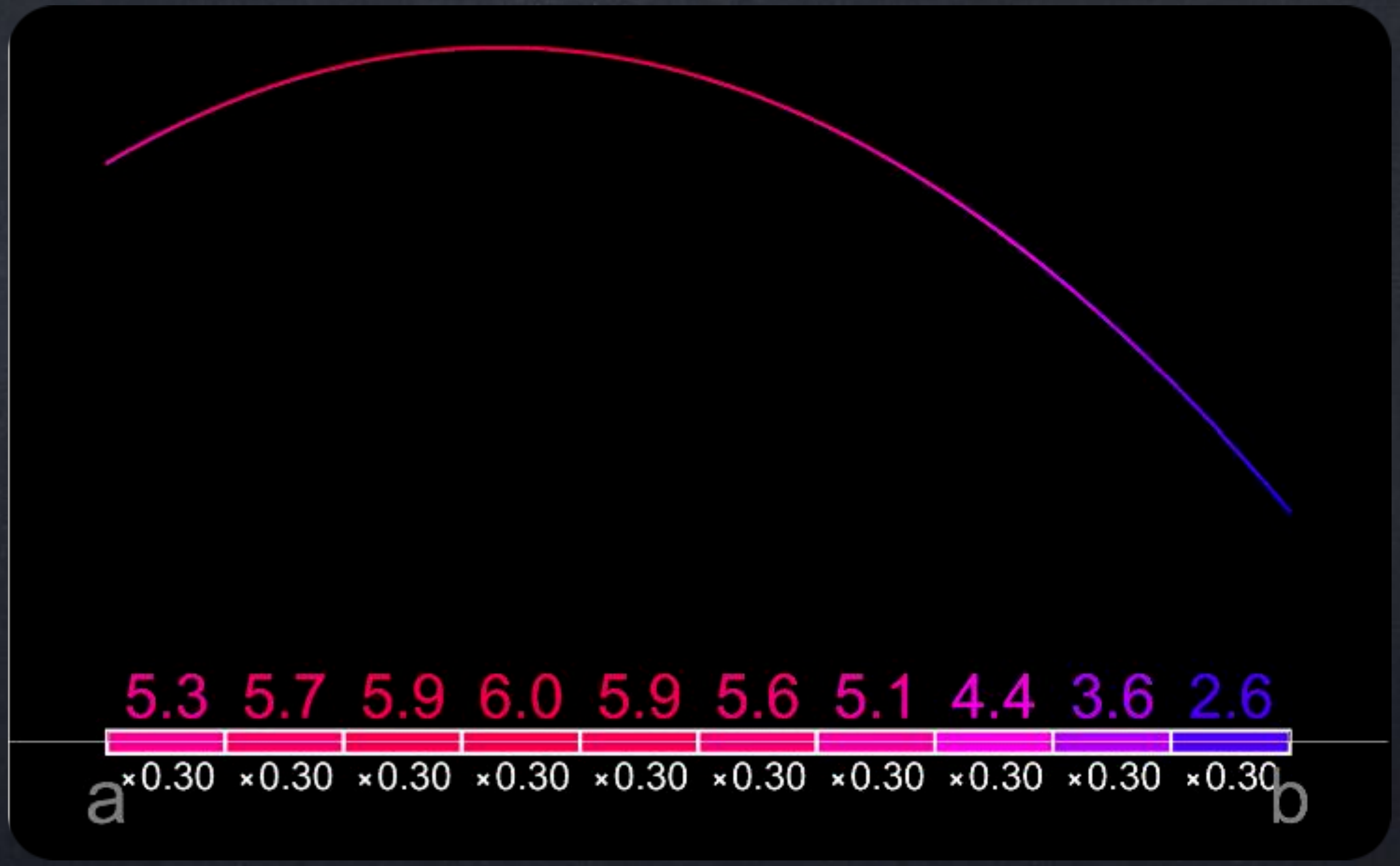
Back students: $\int_2^5 \int_1^3 yx^2 \, dy \, dx = \int_2^5 (4x^2) dx = 156.$

Why are these the same?

Why is this different?

Analysis 1: $\int_a^b f(x)dx$

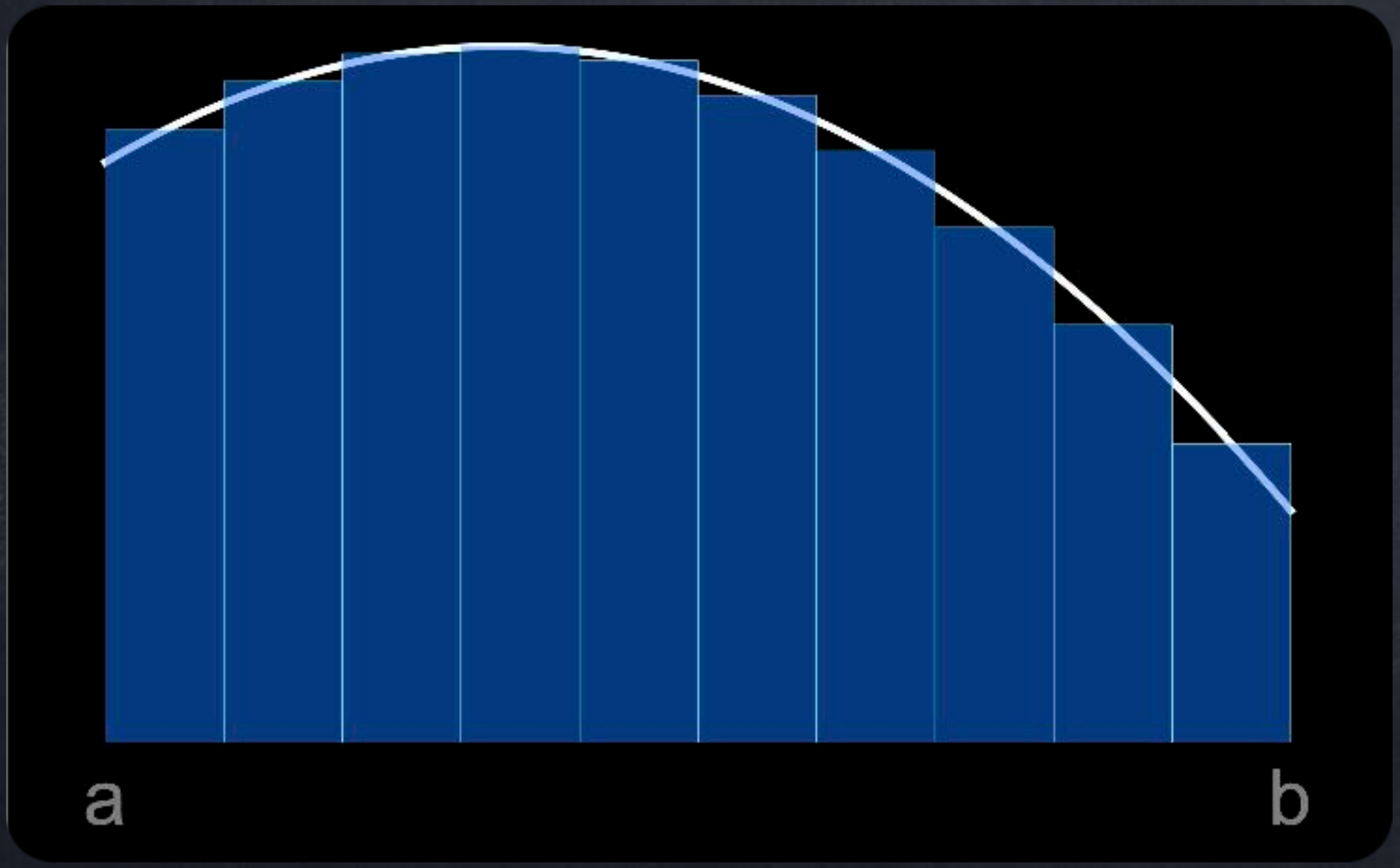




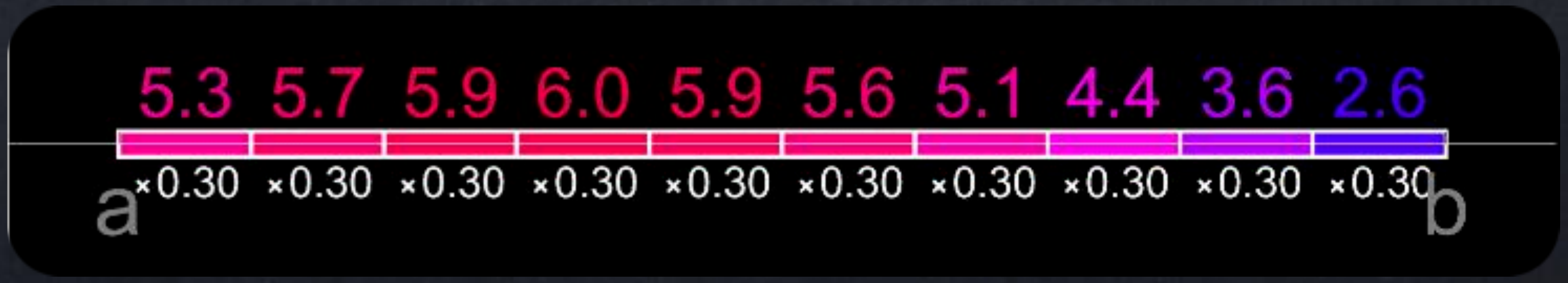
Area

Analysis 1: $\int_a^b f(x)dx$

The advantage of this version is that it is entirely 1D, which is good since $f(x)$ has only 1 input.



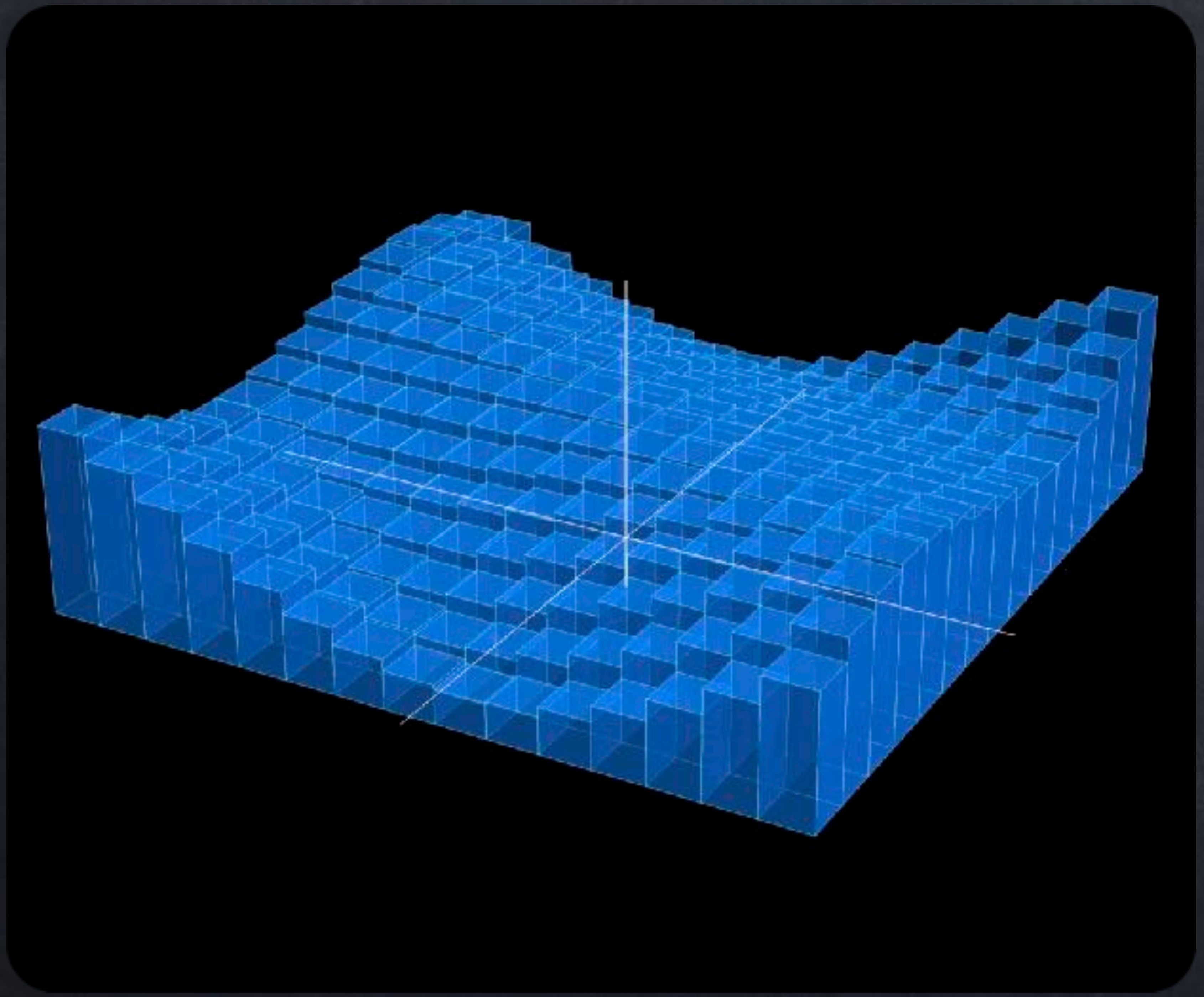
Area



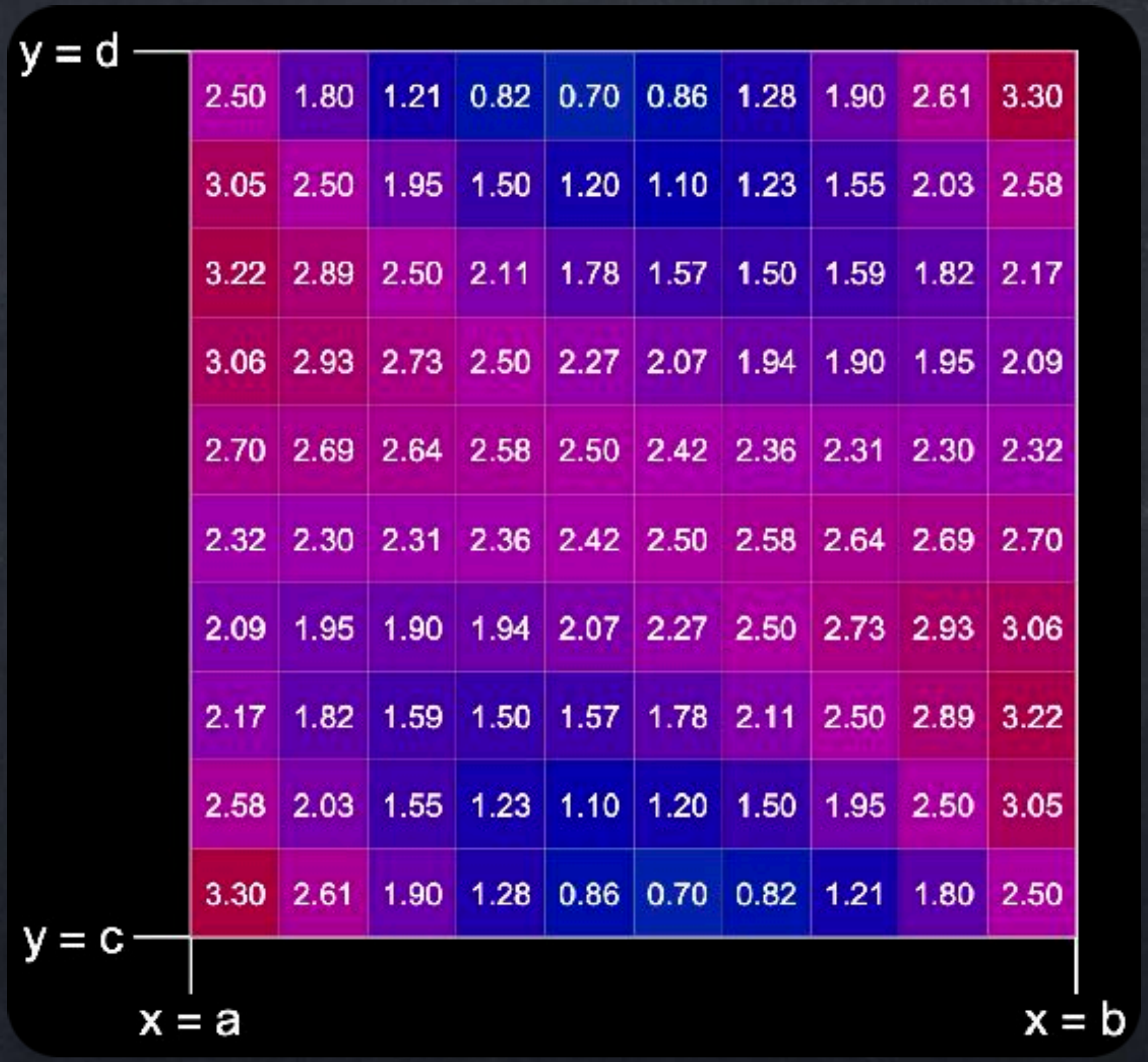
Anything

Analysis 2: $\int_a^b \int_c^d f(x,y)\,dy\,dx$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$



Volume



Anything

$\iint$ over a rectangle

If both the $\int \ldots \mathrm{d}x$ and the $\int \ldots \mathrm{d}y$ have constants (e.g., numbers) for the integral bounds, then the iterated integral describes adding up values of a function over a *rectangular* region.

For example,

"integrate yx^2 over the rectangle $2 \leq x \leq 5$, $1 \leq y \leq 3$"

means to calculate $\int_1^3 \int_2^5 yx^2 \,\mathrm{d}x\,\mathrm{d}y = 156$ from before.

The reason $\int_2^5 \int_1^3 yx^2 \,\mathrm{d}y\,\mathrm{d}x$ is also 156 is because this integral setup describes exactly the *same* xy*-rectangle*.

Although we cannot simply replace $\mathrm{d}x\mathrm{d}y$ by $\mathrm{d}y\mathrm{d}x$ in a double integral, both of these represent a tiny piece of area. For this reason it is common to see "$\mathrm{d}A$" used when writing double integrals.

For example,

"$\displaystyle\iint_R \frac{\sin(y)}{x}\,\mathrm{d}A$, where R is the rectangle with $1 \leq x \leq 9$ and $0 \leq y \leq \pi$."

means exactly the same thing as

$$\text{“}\int_0^{\pi}\int_1^{9} \frac{\sin(y)}{x}\,\mathrm{d}x\mathrm{d}y\text{”}$$

or "$\int_1^9 \int_0^{\pi} \frac{\sin(y)}{x}\,\mathrm{d}y\mathrm{d}x$".

The following tasks are exactly the same:

- Find $\displaystyle\iint_R \frac{\sin(y)}{x}\,\mathrm{d}A$ where R is the rectangle with $1 \le x \le 9$ and $0 \le y \le \pi$.
- Integrate $f(x,y) = \dfrac{\sin(y)}{x}$ over the rectangle with $(1,0)$ as the bottom left corner and $(9,\pi)$ as the top right corner.
- Calculate $\displaystyle\iint_R \frac{\sin(y)}{x}\,\mathrm{d}A$ with $R = \{(x,y) : 1 \le x \le 9,\ 0 \le y \le \pi\}$.
- Evaluate $\displaystyle\iint_R \frac{\sin(y)}{x}\,\mathrm{d}A$ with $R = [1,9] \times [0,\pi]$.
- Find $\displaystyle\int_0^\pi \int_1^9 \frac{\sin(y)}{x}\,\mathrm{d}x\mathrm{d}y$.
- Calculate $\displaystyle\int_1^9 \int_0^\pi \frac{\sin(y)}{x}\,\mathrm{d}y\mathrm{d}x$.

To get the answer, we have to do one of these.

Task: Find $\displaystyle\iint_R \frac{\sin(y)}{x}\,\mathrm{d}A$ where R is the rectangle with $(1,0)$ as the bottom left corner and $(9,\pi)$ as the top right corner.

Answer: 2ln(9)

Volume vs. mass

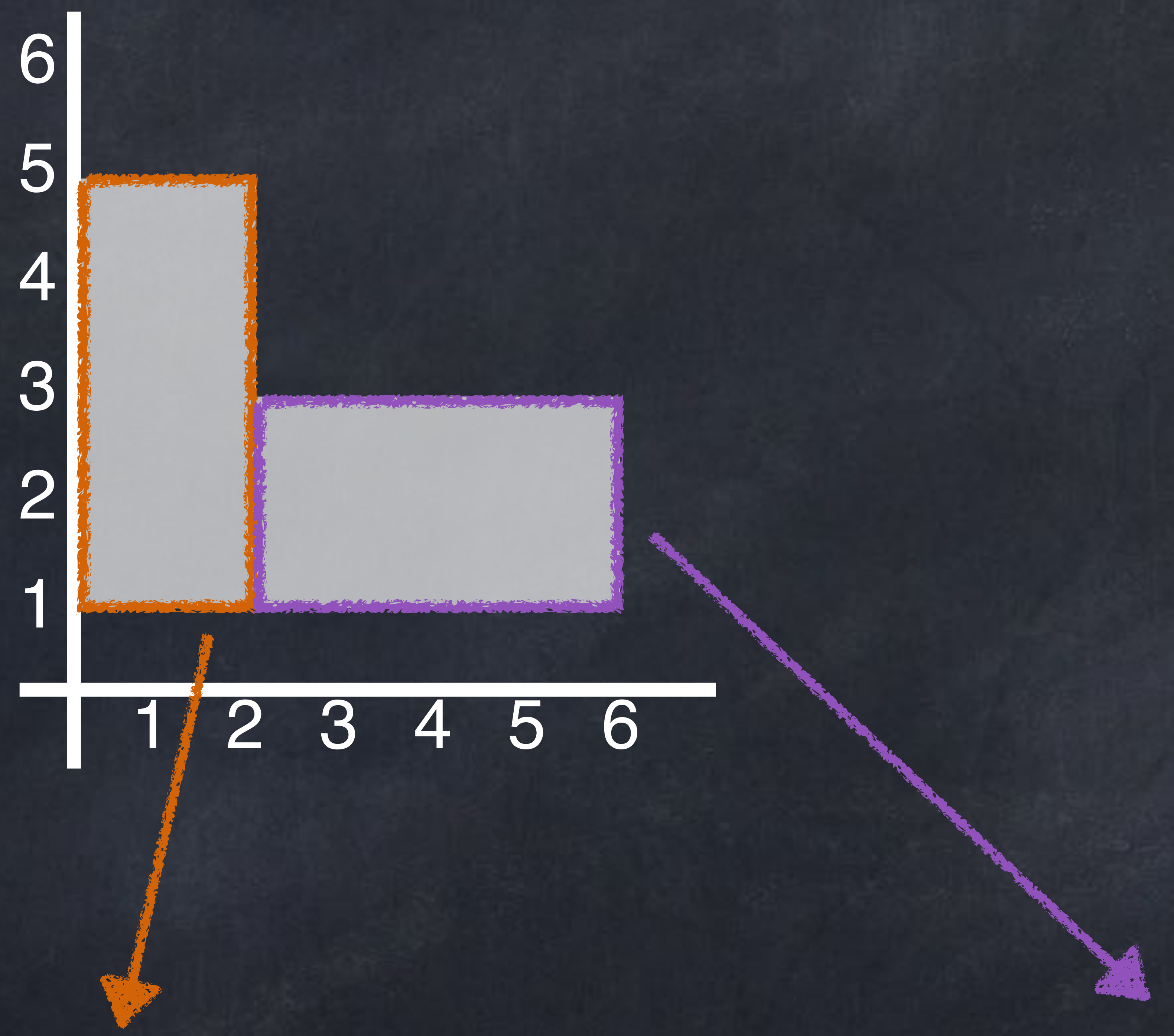
If $f(x, y)$ is thought of as height, then $\iint_R f \, \mathrm{d}A$ calculates volume $(f > 0)$.

If $f(x, y)$ is thought of as density, then $\iint_R f \, \mathrm{d}A$ calculates **mass** $(f > 0)$.

If $f(x, y)$ is thought of as charge density, then $\iint_R f \, \mathrm{d}A$ calculates **total charge**.

- Example: $\int_1^5 \int_1^3 y \, \mathrm{d}y \mathrm{d}x = 16$ is the mass of a rectangle whose density is $f(x, y) = y$ (so it is more dense at the top, less dense at the bottom).

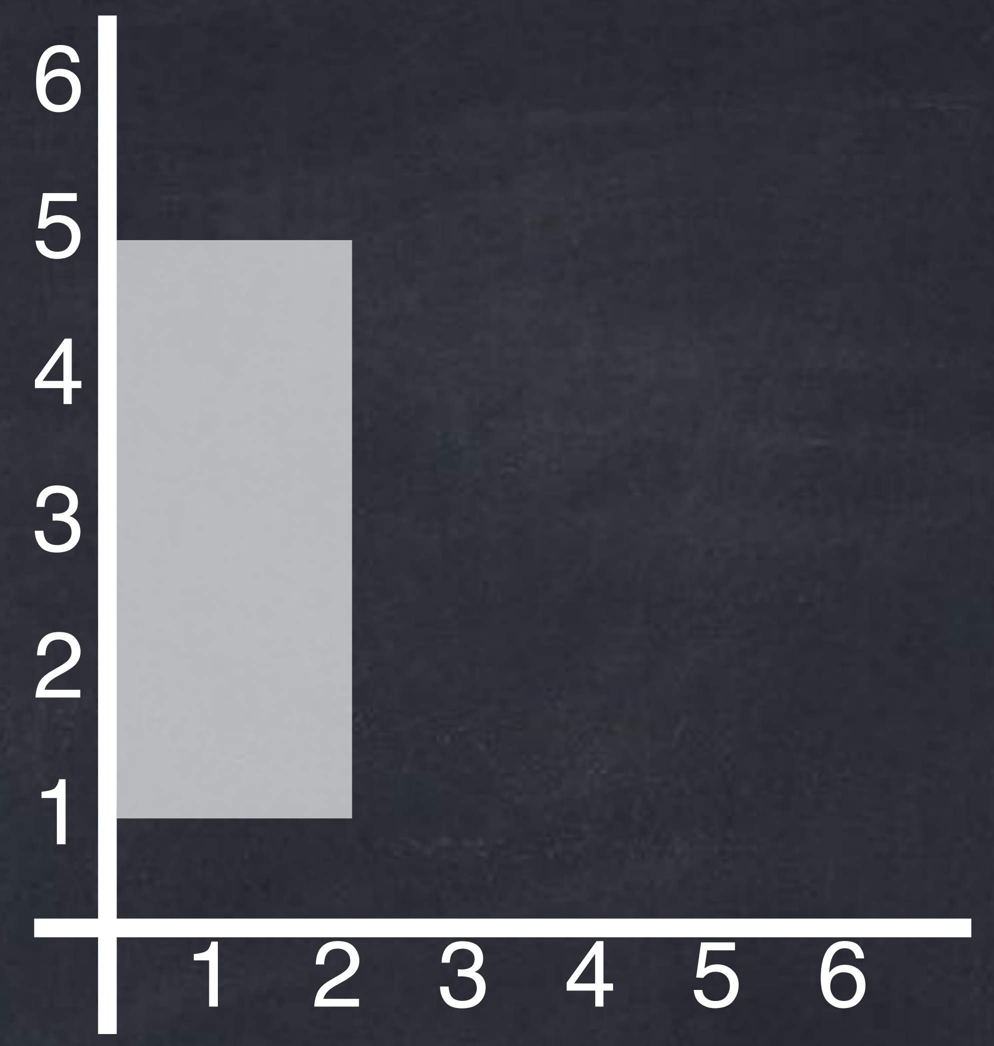
How can we calculate $\iint_D (x+y)\,\mathrm{d}A$ if D is the region below?



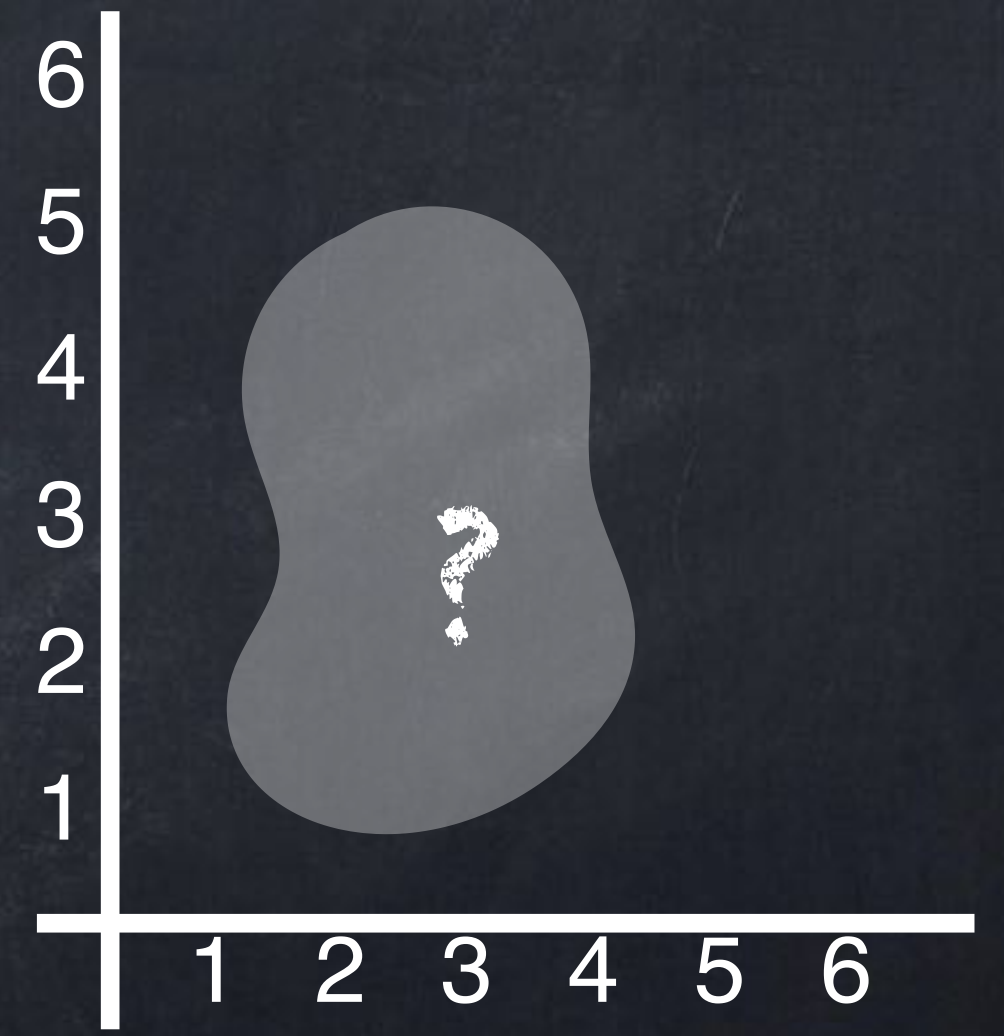
Answer: $\iint_D (x+y)\,\mathrm{d}A = \int_0^2 \int_1^5 (x+y)\mathrm{d}y\mathrm{d}x + \int_2^6 \int_1^3 (x+y)\mathrm{d}y\mathrm{d}x$

or various other sums of two iterated integrals.

Example A: $$\int_0^2 \int_1^5 (x+y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^2 (4x+12)\,\mathrm{d}x = 32.$$

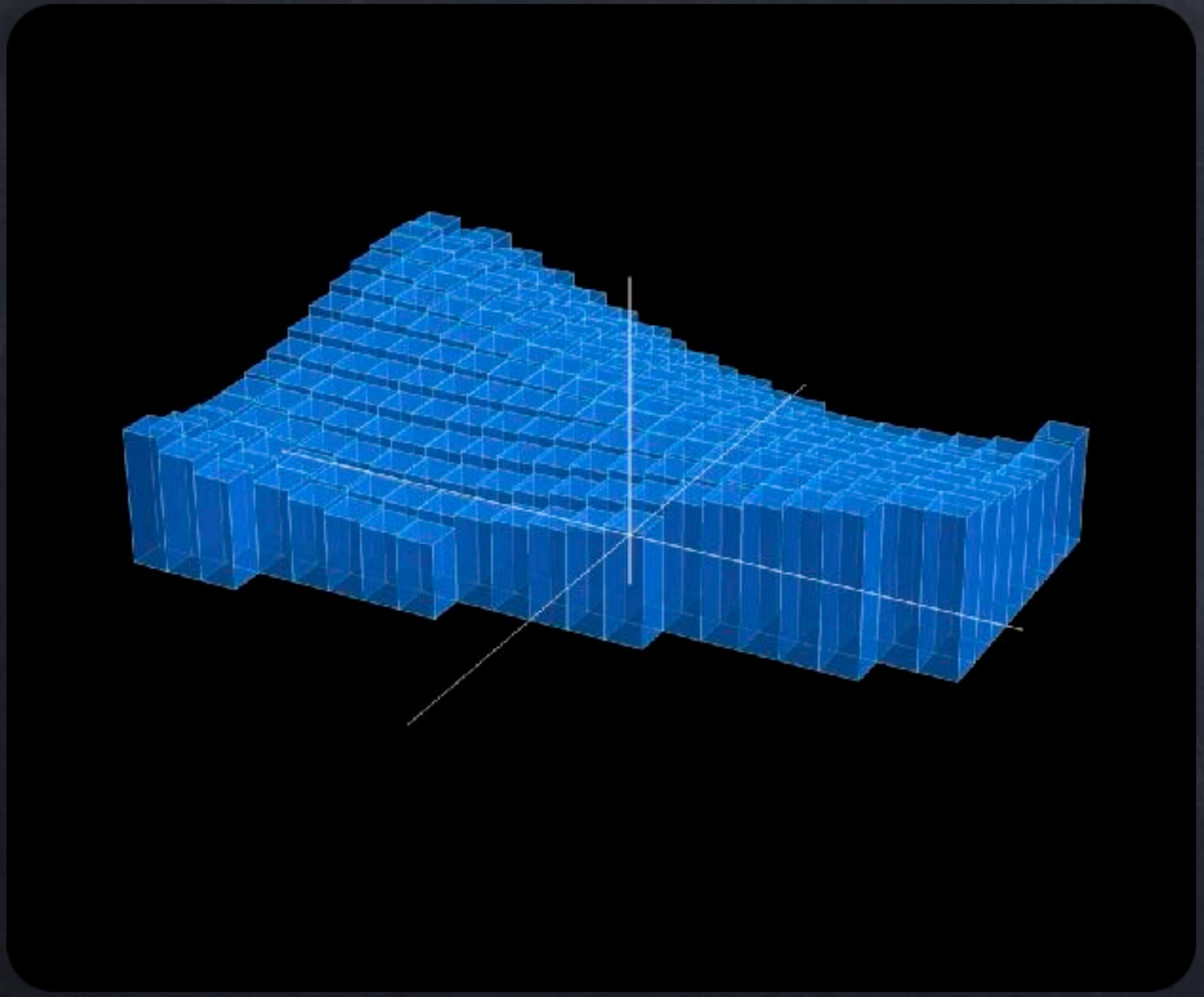


Example B: $$\int_0^2 \int_1^{2x+1} (x+y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^2 \left(4x^2+2x\right)\mathrm{d}x = \frac{16}{3}.$$

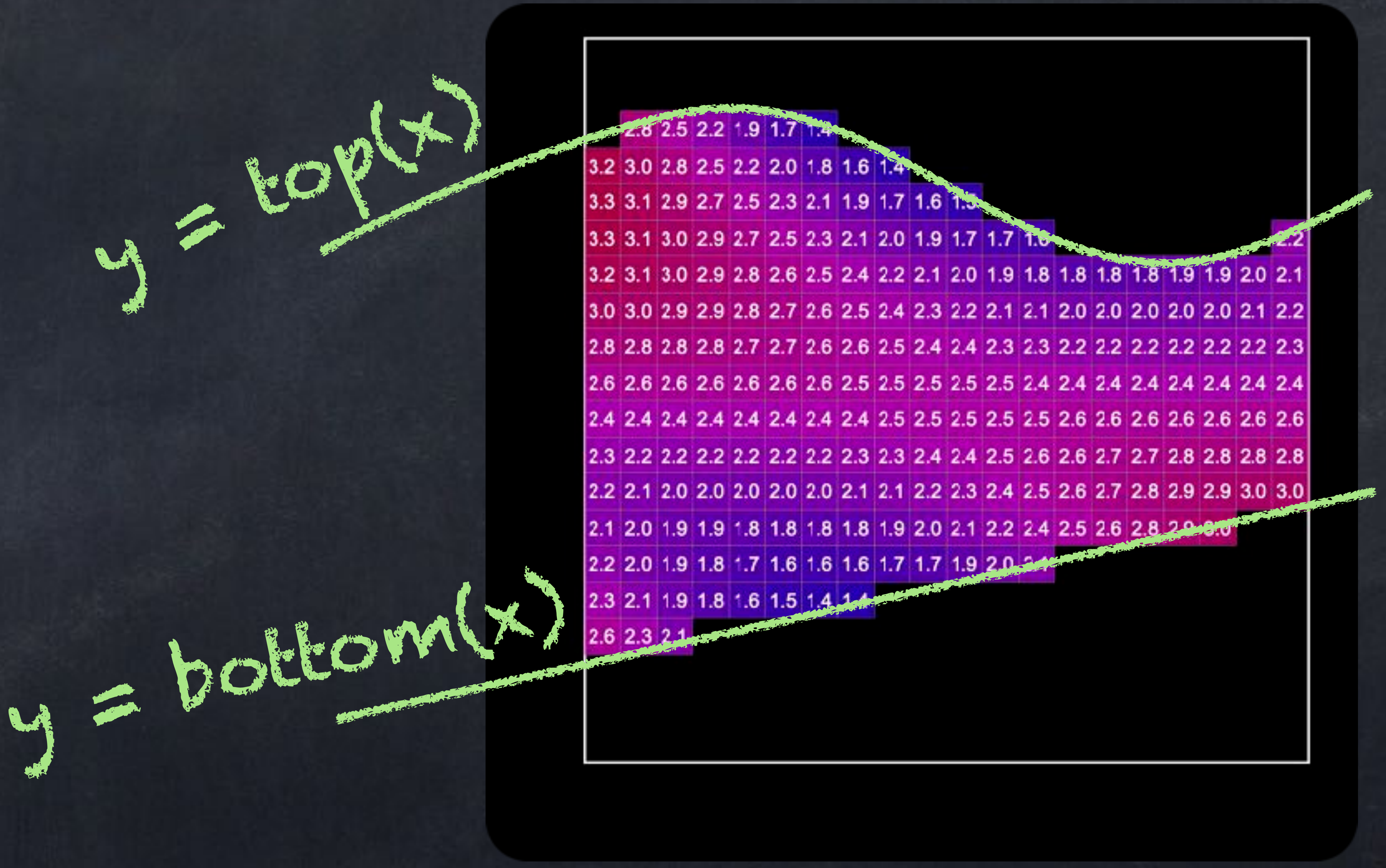


The calculation of Task 2 is not very much harder than Task 2, but what does this new iterated integral *mean*?

Double integral visualization

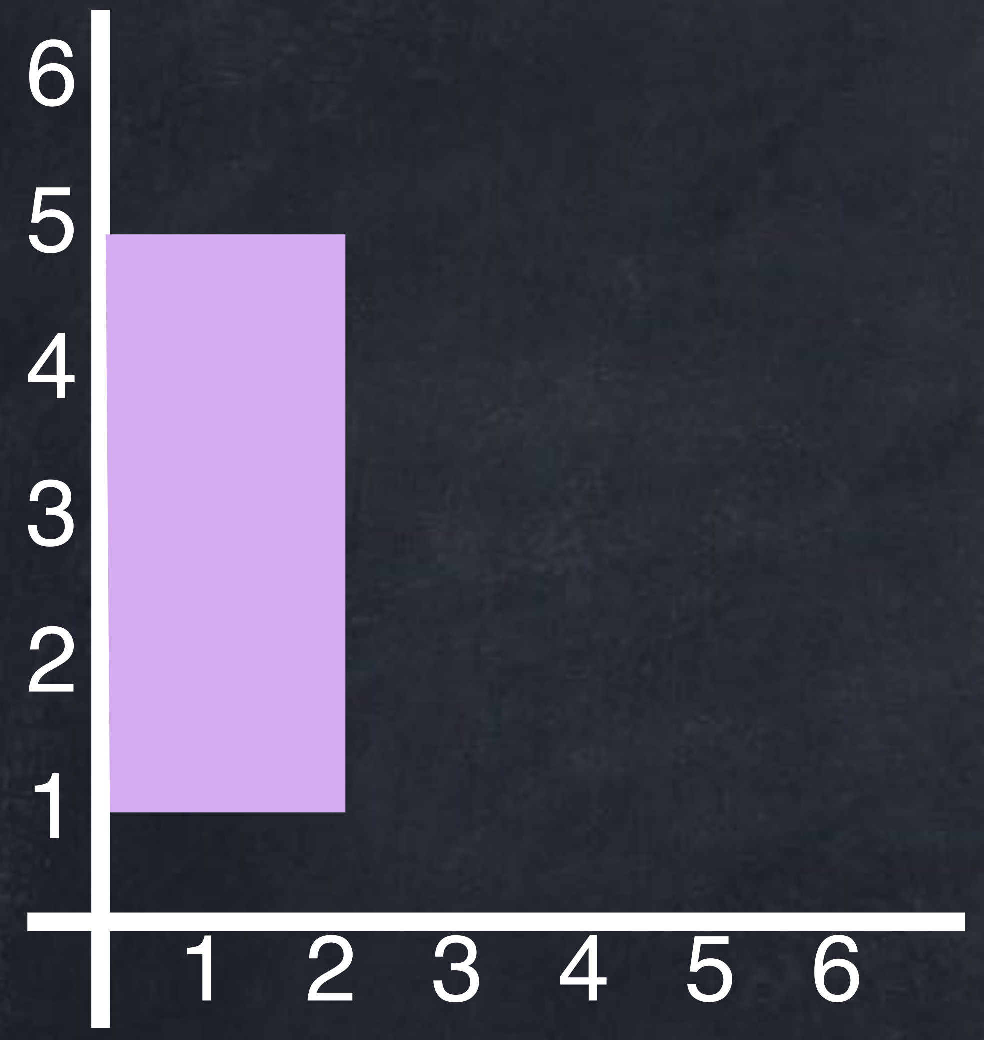


Volume

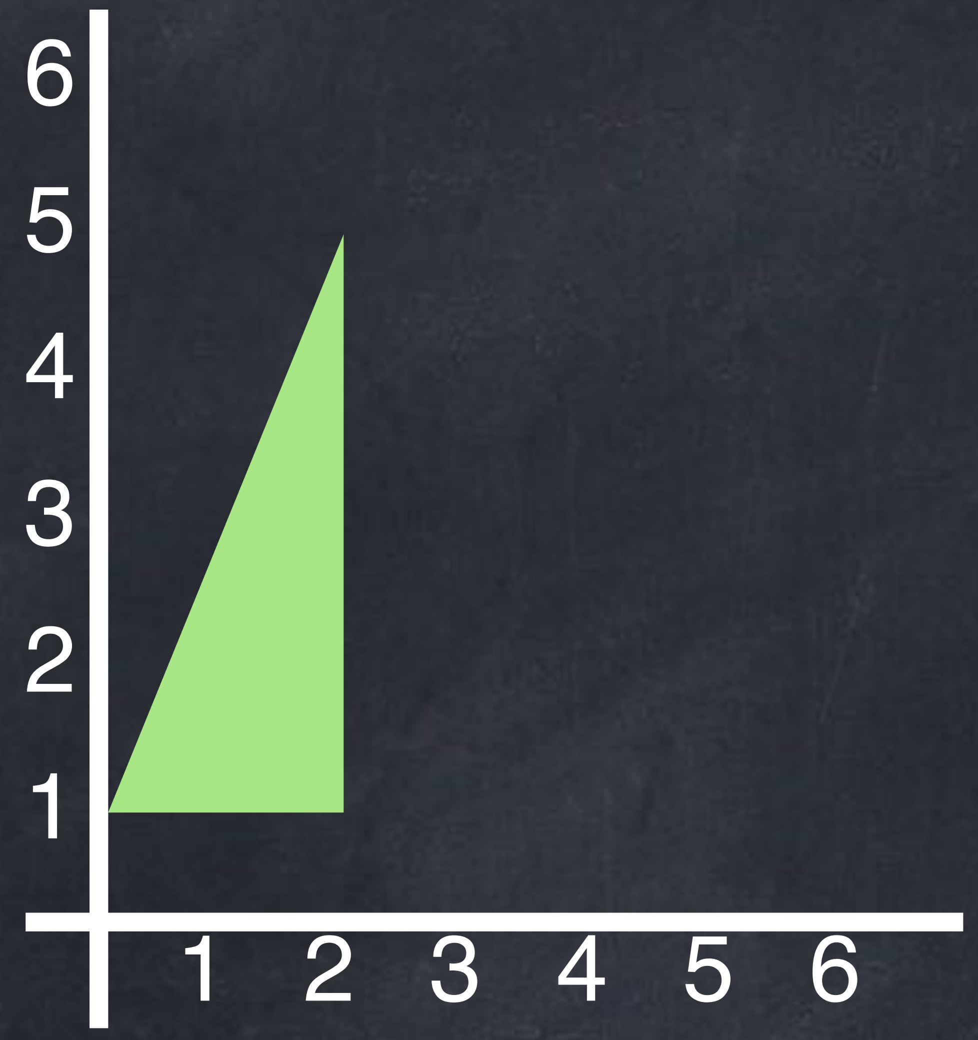


Anything

Ex. A used $\int_0^2 \int_1^5 (x+y)\mathrm{d}y\mathrm{d}x.$



Ex. B used $\int_0^2 \int_1^{2x+1} (x+y)\mathrm{d}y\mathrm{d}x.$



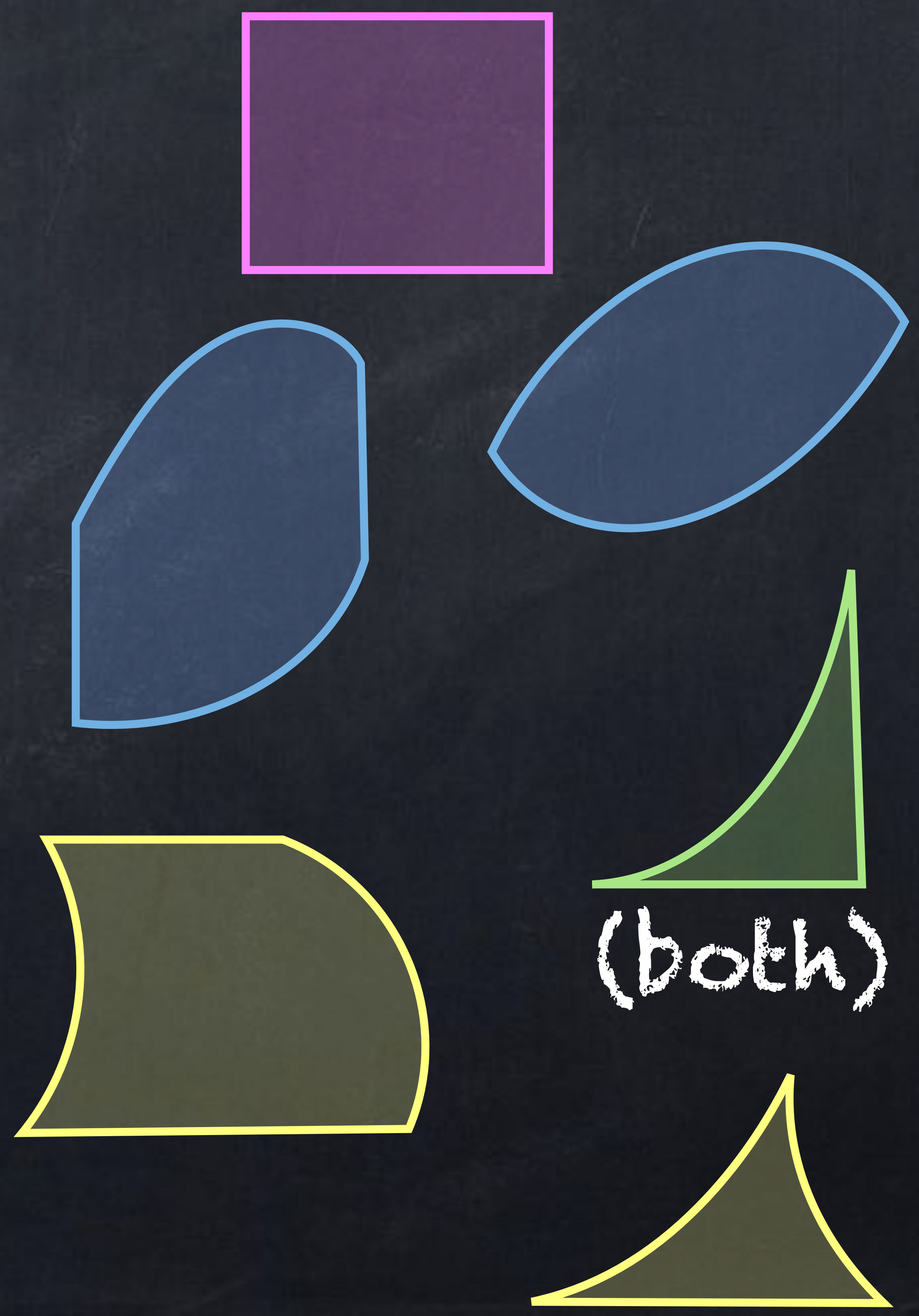
Rectangles and triangles are both very common regions for double-integrals.

What other regions can we use?

Different kinds of regions

The way we write an iterated integral for $\iint_D f\,\mathrm{d}A$ depends on the shape of the region D.

- Rectangle: $\displaystyle\int_{\text{left}}^{\text{right}}\int_{\text{bot.}}^{\text{top}} f\,\mathrm{d}y\,\mathrm{d}x$ or $\displaystyle\int_{\text{bot.}}^{\text{top}}\int_{\text{left}}^{\text{right}} f\,\mathrm{d}x\,\mathrm{d}y$
- L/R sides are walls (or points): $\displaystyle\int_{\text{left}}^{\text{right}}\int_{\text{bottom fn.}}^{\text{top function}} f\,\mathrm{d}y\,\mathrm{d}x$
- Top and bottom are flat (or points): $\displaystyle\int_{\text{bot.}}^{\text{top}}\int_{\text{left fn.}}^{\text{right fn.}} f\,\mathrm{d}x\,\mathrm{d}y$



Different kinds of regions

The way we write an iterated integral for $\iint_D f \, \mathrm{d}A$ depends on the shape of the region D.

There are two other common region shapes that we will *not* be using in this course (but you might see them in other classes in the future):

- Pie slice, ring, other piece of a disk: $\displaystyle\int_{\text{start angle}}^{\text{stop angle}} \int_{\text{inner rad.}}^{\text{outer rad.}} \underbrace{f\, r \,\mathrm{d}r\, \mathrm{d}\theta}_{\mathrm{d}A}$

- Region bounded by level curves: $\displaystyle\int_a^b \int_c^d \underbrace{f \;\; J \;\; \mathrm{d}u\, \mathrm{d}v}_{\mathrm{d}A}$

- The expressions

$$\int_0^1 \int_2^8 x^3 \, \mathrm{d}y \, \mathrm{d}x \qquad \int_1^4 \int_v^{v+1} u \, \mathrm{d}u \, \mathrm{d}v \qquad \int_2^e \int_{\sin(y)}^{3y} x^y \, \mathrm{d}y \, \mathrm{d}x$$

  are all examples of **iterated integrals** and **double integrals** (both).

- Expressions like $\iint_R x\cos(y) \, \mathrm{d}A$ are also **double integrals** (but not iterated integrals).

However, I will often use "double" and "iterated" interchangeably.

- Iterated integrals can also be used for triple integrals $\iiint_R f \, \mathrm{d}V$ in 3D, but those are not part of this course.

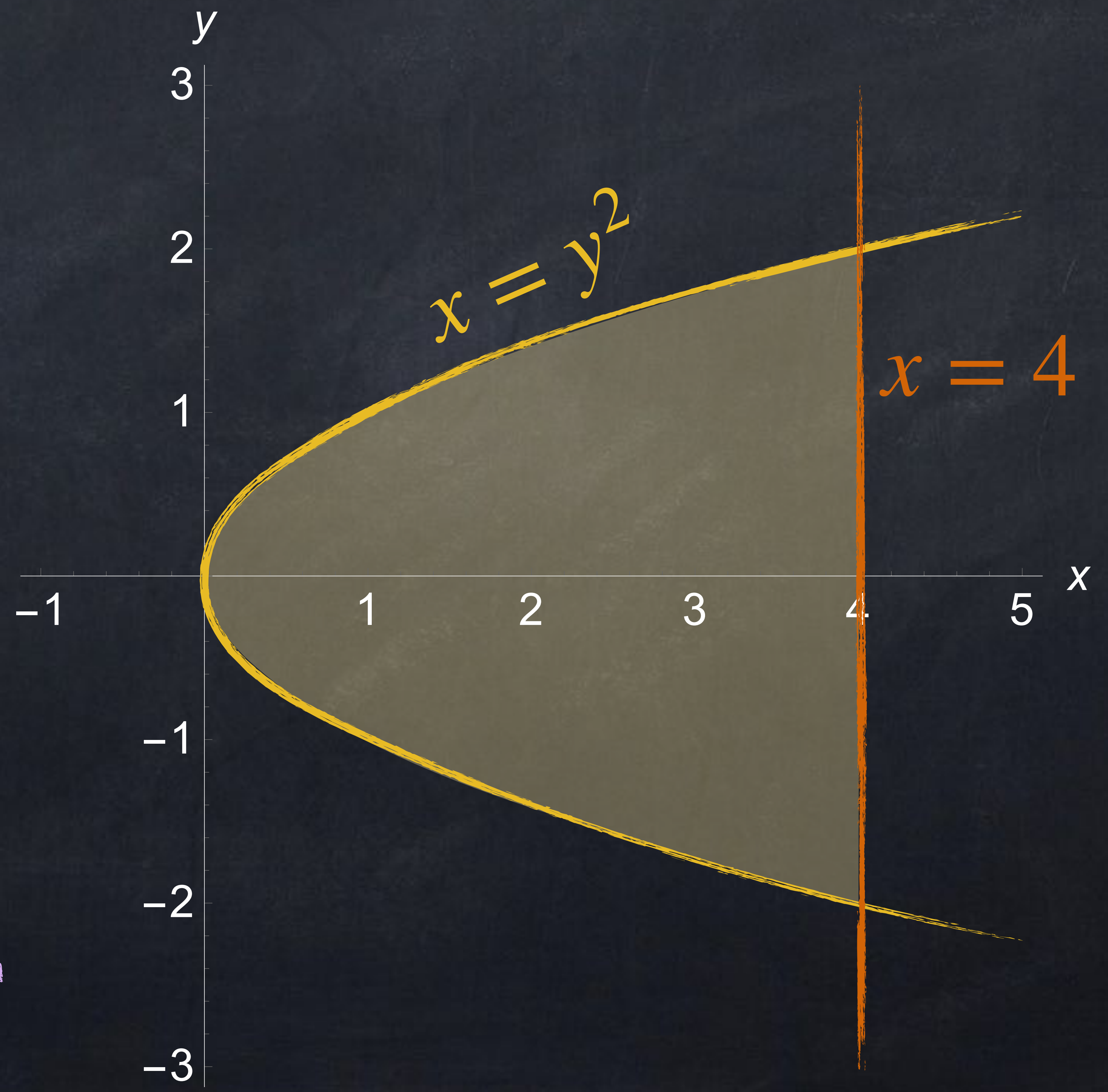
Example: Find $\iint_R \frac{3}{4} e^{(x^{3/2})}\, \mathrm{d}A$ with R bounded by $y^2 = x$ and $x = 4$.

This could also be written “integrate $\frac{3}{4} e^{(x^{3/2})}$ over the region bounded by …”.

Step 0: Draw the region.

Step 1: Write an iterated integral.

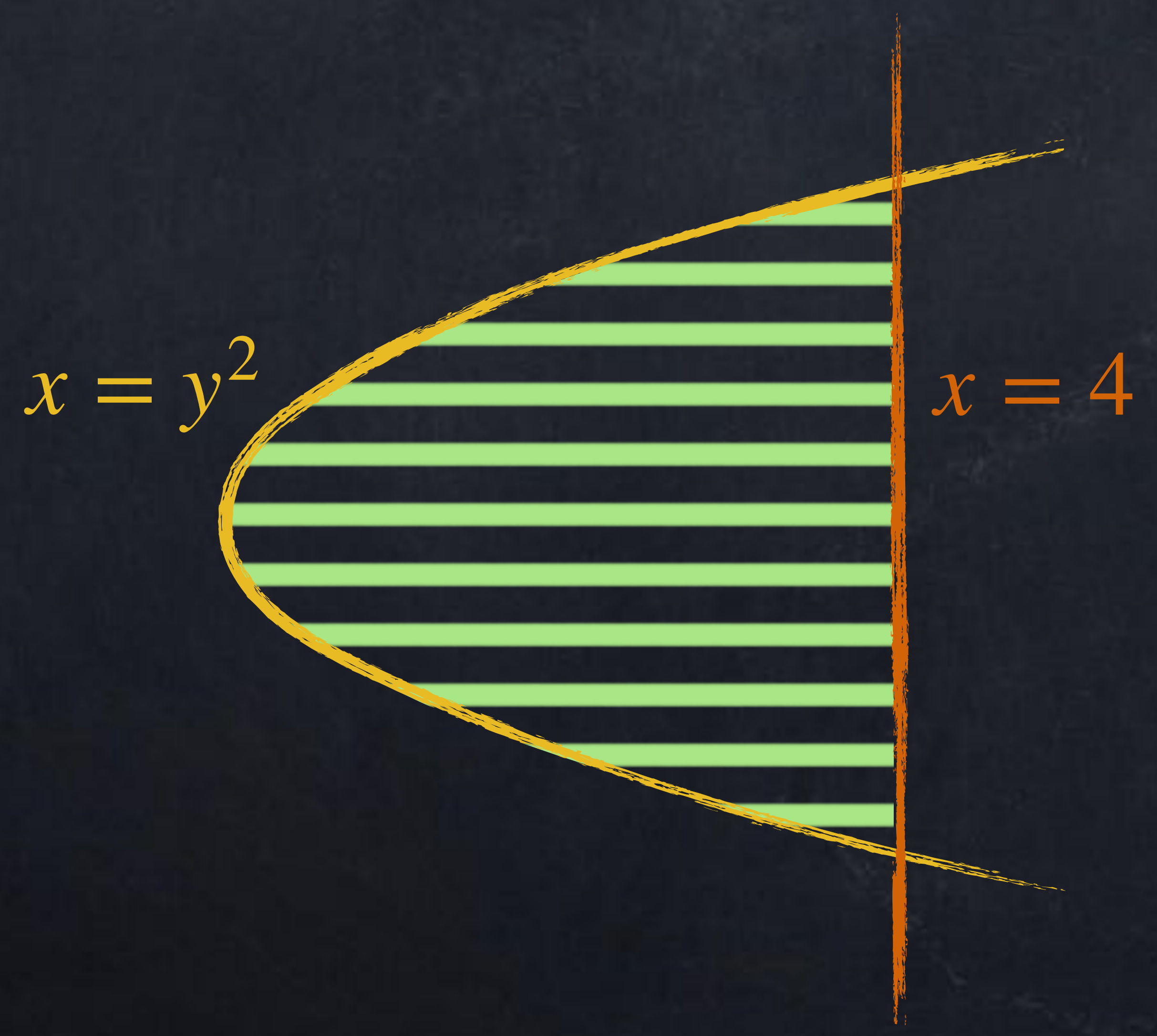
Step 2&3: Evaluate the inside integral (2), then the outside (3).

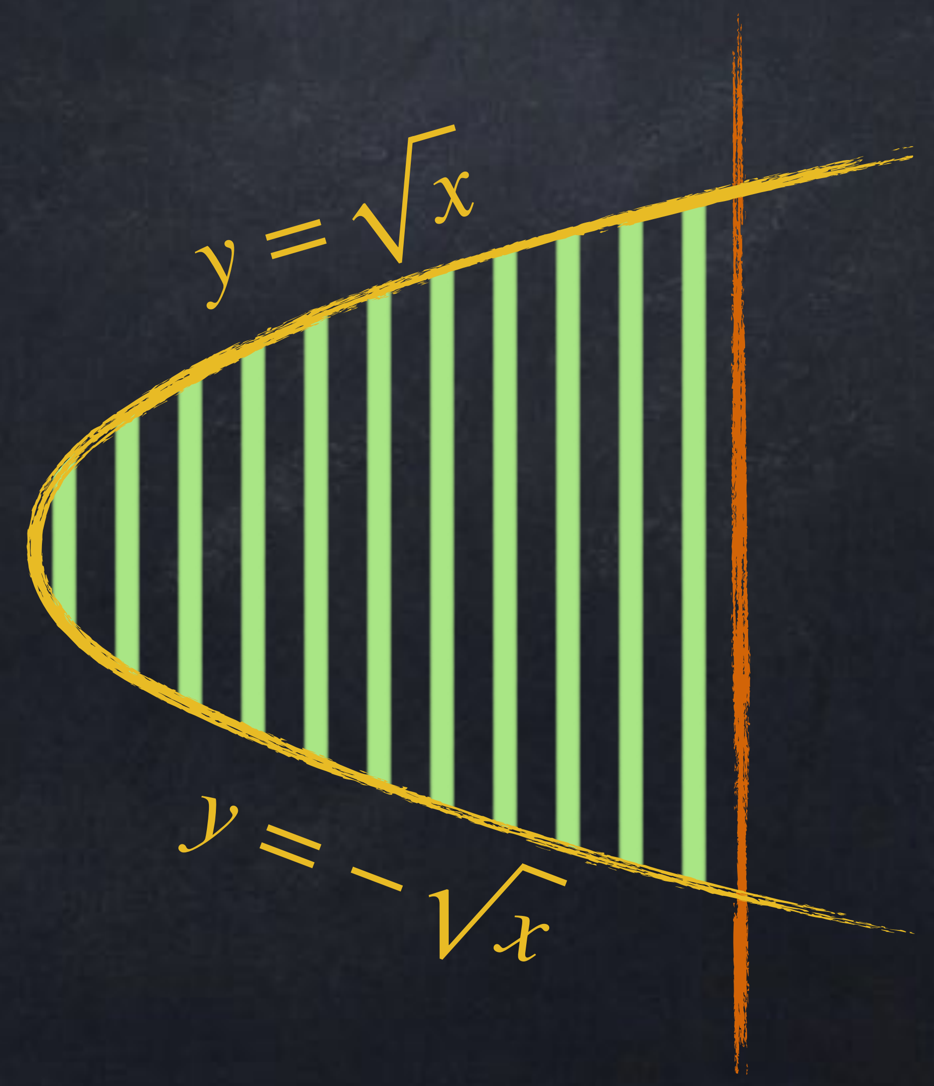


Example: Find $\iint_R \frac{3}{4} e^{(x^{3/2})}\, dA$ with R bounded by $y^2 = x$ and $x = 4$.

Step 1: Write an iterated integral.

$$\int_{-2}^{2}\int_{y^2}^{4} \frac{3}{4} e^{x^{3/2}}\, dx\, dy \quad \text{or} \quad \int_{0}^{4}\int_{-\sqrt{x}}^{\sqrt{x}} \frac{3}{4} e^{x^{3/2}}\, dy\, dx$$



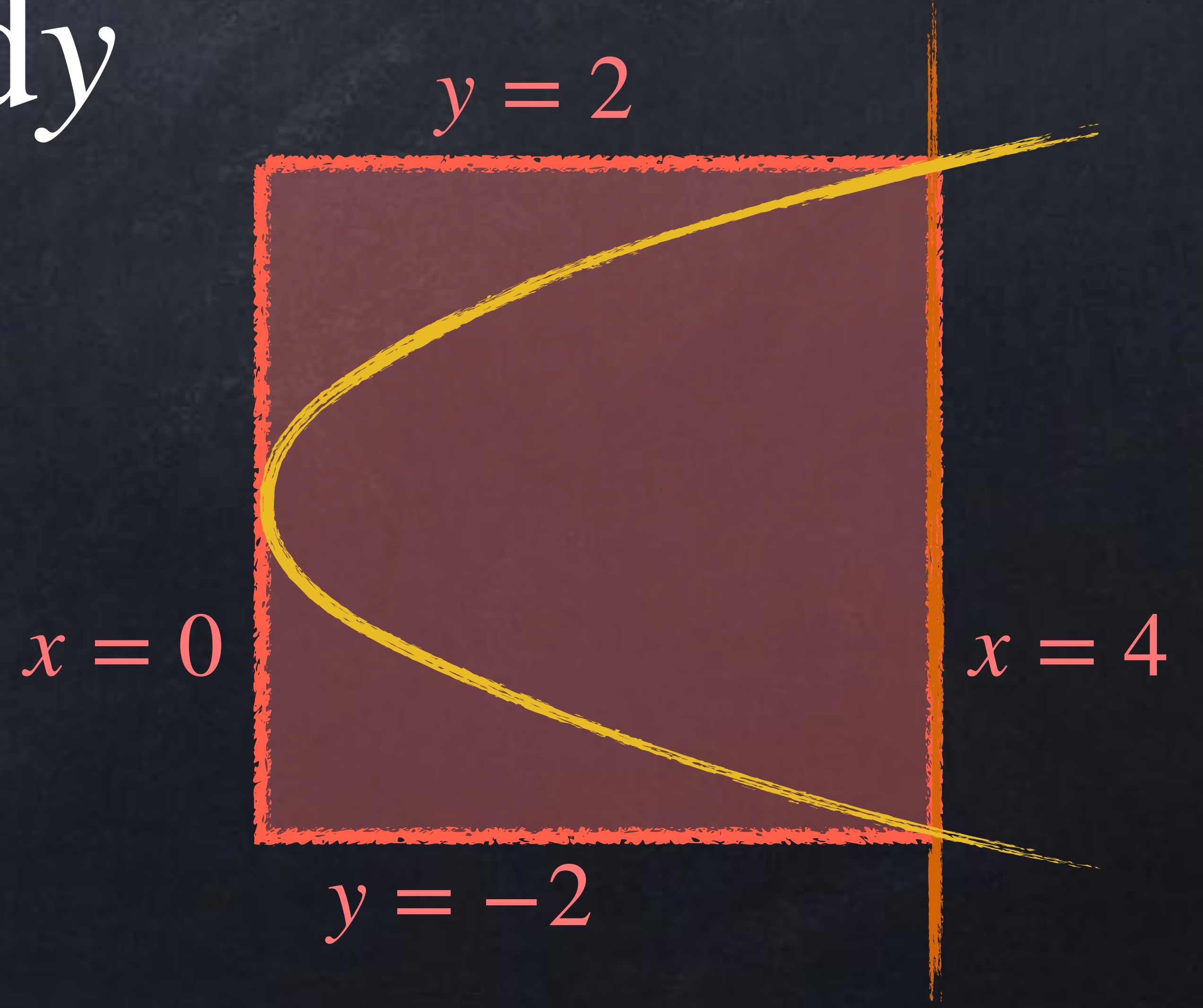


Example: Find $\iint_R \frac{3}{4} e^{(x^{3/2})}\, dA$ with R bounded by $y^2 = x$ and $x = 4$.

Step 1: Write an iterated integral.

$$\int_{-2}^{2}\int_{y^2}^{4} \frac{3}{4} e^{x^{3/2}}\, dx\, dy \quad \text{or} \quad \int_{0}^{4}\int_{-\sqrt{x}}^{\sqrt{x}} \frac{3}{4} e^{x^{3/2}}\, dy\, dx$$

Note: This cannot be $\int_{-2}^{2}\int_{0}^{4} \frac{3}{4} e^{x^{3/2}} dxdy$ because $\int_{-2}^{2}\int_{0}^{4} \ldots dxdy$ describes a rectangle ($\int_{0}^{4}\int_{-2}^{2} \frac{3}{4} e^{x^{3/2}} dydx$ is also a rectangle).



Problem Session

Example: Find $\iint_R \frac{3}{4}e^{(x^{3/2})}\,\mathrm{d}A$ with R bounded by $y^2 = x$ and $x = 4$.

Step 1: Write an iterated integral.

$$\int_{-2}^{2}\int_{y^2}^{4} \frac{3}{4}e^{x^{3/2}}\,\mathrm{d}x\,\mathrm{d}y \quad \text{or} \quad \int_{0}^{4}\int_{-\sqrt{x}}^{\sqrt{x}} \frac{3}{4}e^{x^{3/2}}\,\mathrm{d}y\,\mathrm{d}x$$

Step 2: Evaluate, starting with the inside integral.

~~$$\int_{y^2}^{4} \frac{3}{4}e^{x^{3/2}}\,\mathrm{d}x$$~~ or $$\int_{-\sqrt{x}}^{\sqrt{x}} \frac{3}{4}e^{x^{3/2}}\,\mathrm{d}y$$

There is no formula with $(?)'_x = e^{x^{3/2}}$.

Problem Session

Example: Find $\iint_R \frac{3}{4}e^{(x^{3/2})}\,\mathrm{d}A$ with R bounded by $y^2 = x$ and $x = 4$.

Step 1: Write an iterated integral. $\int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} \frac{3}{4}e^{x^{3/2}}\mathrm{d}y\mathrm{d}x$

Step 2: Evaluate, starting with the inside integral.

$$\int_{-\sqrt{x}}^{\sqrt{x}} \frac{3}{4}e^{x^{3/2}}\mathrm{d}y = \frac{3}{4}e^{x^{3/2}}y\Big|_{y=-\sqrt{x}}^{y=\sqrt{x}} = \frac{3}{4}e^{x^{3/2}}\cdot 2\sqrt{x} = e^{x^{3/2}}\cdot\frac{3}{2}x^{1/2}$$

Outside integral. Using $u = x^{3/2}$,

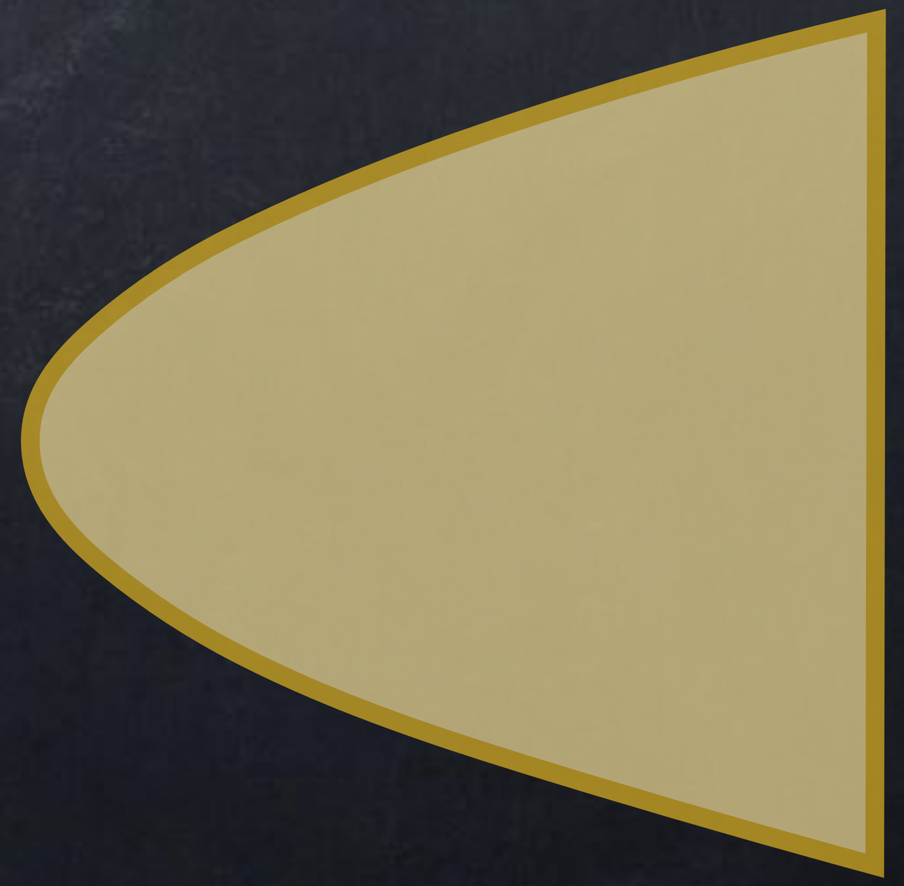
$$\int_0^4 e^{x^{3/2}}\cdot\frac{3}{2}x^{1/2}\mathrm{d}x = \int_0^8 e^u\mathrm{d}u = e^u\Big|_{u=0}^{u=8} = \boxed{e^8 - 1}$$

"Reversing" a $\iint$

Since there is no formula for $\int \frac{3}{4} e^{x^{3/2}} \mathrm{d}x$, if you have to find $\int_{-2}^{2} \int_{y^2}^{4} \frac{3}{4} e^{x^{3/2}} \mathrm{d}x\mathrm{d}y$ by hand, the only way to do this is to use the fact that

$$\int_{-2}^{2} \int_{y^2}^{4} \frac{3}{4} e^{x^{3/2}} \mathrm{d}x\mathrm{d}y = \iint_D \frac{3}{4} e^{x^{3/2}} \mathrm{d}A = \int_{0}^{4} \int_{-\sqrt{x}}^{\sqrt{x}} \frac{3}{4} e^{x^{3/2}} \mathrm{d}y\mathrm{d}x$$

and evaluate the $\mathrm{d}y\mathrm{d}x$ version.



This is called "reversing the order of integration".