Analysis 2

24 April 2024

Warm-up: Describe all functions $f(x)$ for which $f'(x) = 18x^8$.

Last week

Task: Describe all functions $f(x, y)$ for which $\nabla f = \begin{bmatrix} y^2 - 3x^2 \\ 2xy + 1 \end{bmatrix}$.

$f(x,y)$ will have the format

$$\int (2xy+1)\,dy = xy^2 + y + g(x)$$

for some yet-unknown function $g(x)$.

From the formula above, $f'_x = y^2 + 0 + g'(x)$.

But from ∇f we also know $f'_x = y^2 - 3x^2$.

So it must be that $g' = -3x^2$. Thus $g(x) = -x^3 + C$, and

$$\boxed{f = xy^2 + y - x^3 + C}$$

Last week

Task: Describe all functions $f(x, y)$ for which $\nabla f = \begin{bmatrix} y^2 - 3x^2 \\ 2xy + 1 \end{bmatrix}$.

$$f = \int (2xy+1)\,dy = xy^2 + y + g(x)$$

$$f'_x = y^2 + 0 + g'(x) = y^2 - 3x^2$$

$$g' = -3x^2$$

$$\boxed{f = xy^2 + y - x^3 + C}$$

	a scalar (number) as output	a vector (or multiple numbers) as output
a scalar (number) as input	$f(x)$ $x(t)$ $P(x)$	"vector function" $\vec{r}(t)$
a vector (or multiple numbers) as input	"scalar field" $f(x, y)$ $V(x, y, z, t)$	"vector field" $\overrightarrow{F}(x, y, z)$

Vector fields

A **vector field** is any function $f : \mathbb{R}^n \to \mathbb{R}^m$ with $n > 1$ and $m > 1$.

They are extremely useful in physics (e.g., electric field).

Examples:

$$\overrightarrow{F}(x, y) = \begin{bmatrix} y^2 - 3x^2 \\ 2xy + 1 \end{bmatrix}$$

$$\overrightarrow{E}(x, y, z, t) = \begin{bmatrix} 2xe^{-t} \\ 2ye^{-t} \\ 2ze^{-t} \end{bmatrix}$$

Vector fields

The **potential** of a vector field $\overrightarrow{F}(x, y)$ is a scalar field $f(x, y)$ for which

$$\nabla f = \overrightarrow{F}.$$

Some vector fields don't have any potential.

- For $\overrightarrow{G}(x, y) = \begin{bmatrix} 2x\cos(y) \\ x^2\sin(y) \end{bmatrix}$, does any potential exist?
- For $\overrightarrow{K}(x, y) = \begin{bmatrix} 2x\sin(y) \\ x^2\cos(y) \end{bmatrix}$, does any potential exist?

Vector fields

The **potential** of a vector field $\overrightarrow{F}(x, y)$ is a scalar field $f(x, y)$ for which

$$\nabla f = \overrightarrow{F}.$$

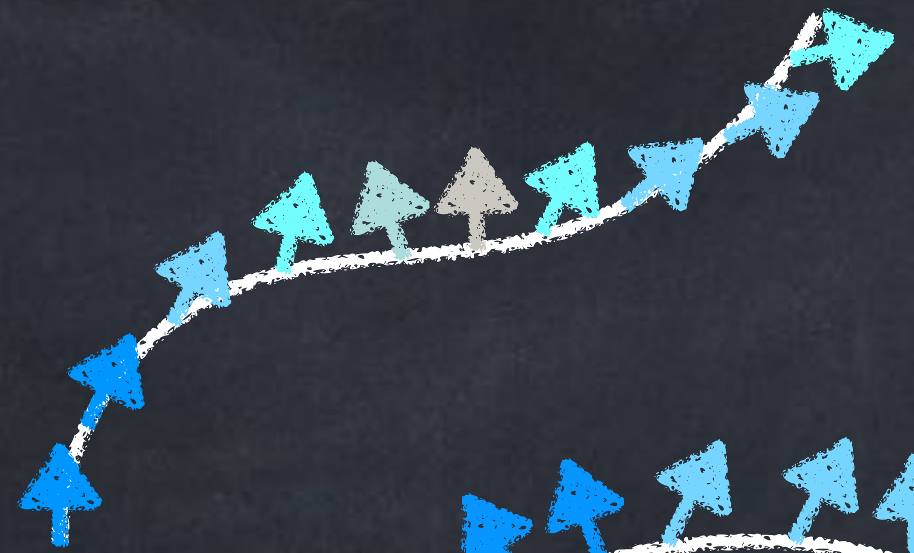
Some vector fields don't have any potential.

The vector field $\overrightarrow{F}(x, y) = \begin{bmatrix} M(x, y) \\ N(x, y) \end{bmatrix}$ has a potential if and only if $M'_y = N'_x$.

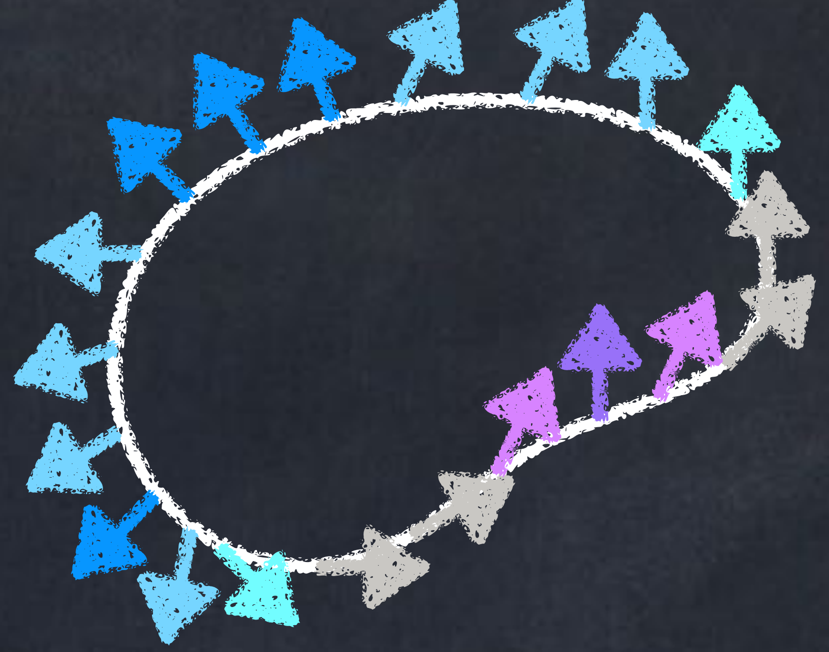
This checks that $f''_{xy} = f''_{yx}$

For $\overrightarrow{F}(x, y) = M(x, y)\hat{\imath} + N(x, y)\hat{\jmath}$, there are…

- path integrals $\displaystyle\int_C \overrightarrow{F} \cdot \mathrm{d}\vec{r}$



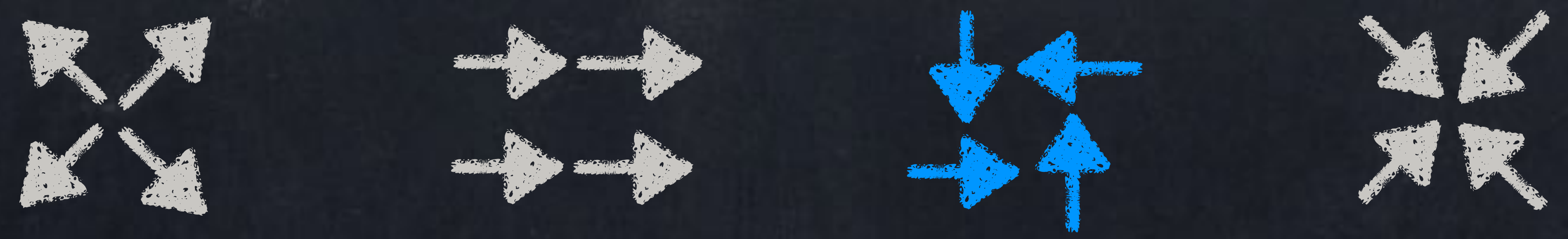
- flux integrals $\displaystyle\oint_C \overrightarrow{F} \cdot \vec{n}\, \mathrm{d}s$



positive
zero
negative

- divergence $\nabla \cdot \overrightarrow{F}$

- curl $\nabla \times \overrightarrow{F}$



but we will not be talking about any of these 😕

Math 1510

These are the main topics we will cover this year:

Scalar fields

- Path integrals
- Gradients
- Critical points
- Double integrals

Vector fields

- Potential

Differential equations

- Vocabulary
- Separable
- Exact
- First-order linear
- Higher-order linear

An **ordinary differential equation** (**ODE**) is an equation that includes a *derivative* of a function with one variable.

When a task says "Solve the differential equation…", it means to find a formula with that function that does *not* use derivatives.

- Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 18x^8$.
- Solve $y' = x$.
- Solve $y' = y$.
- Solve $\sin(y' + x) = \ln(x - y)$.

A **partial differential equation** (**PDE**) is an equation that includes a *partial derivative* of a function with *multiple* variables.

Examples:

- $\dfrac{\partial f}{\partial x} = 18x^8$
- $f_t' = 5f_{xx}''$

The only PDEs we will do in this class will be finding potential.

- Task: Solve $f_x' = y^2e^{xy},\ f_y' = xye^{xy} + e^{xy}$.

Task: Solve $f'_x = y^2 e^{xy},\ f'_y = xye^{xy} + e^{xy}$.

Many differential equations come from physical setups.
You do <u>not</u> need to know these names.

- Heat: $u'_t = \alpha\, u''_{xx}$
- Wave: $f''_{tt} = v^2 f''_{xx}$
- Laplace: $f''_{xx} + f''_{yy} = 0$
- Free-fall: $y'' = -g$
- Unrestricted growth: $P' = r\,P$
- Undamped oscillator: $m\,x'' = -k\,x$
- Damped oscillator: $x'' + \beta x' + \gamma x = 0$
- Pendulum: $Ly'' + G\sin(y) = 0$
- RC circuit: $\dfrac{\mathrm{d}V}{\mathrm{d}t} = R\dfrac{\mathrm{d}I}{\mathrm{d}t} + \dfrac{1}{C}I$

In this class, our PDEs will *always* be

- $f'_x = \_\_\_\_\_, \; f'_y = \_\_\_\_\_.$

Our ODEs could be very different.

- $f' = 3x\sin(x^2)$
- $f' = x^3 f$
- $f'' + 2f' + 5f = \sin(x)$

We will learn how to solve all three of these before the end of the semester.

	a scalar (number) as output	a vector (or multiple numbers) as output
a scalar (number) as input	$f(x)$ $x(t)$ $y(x)$ $y(t)$	$\vec{r}(t)$
a vector (or multiple numbers) as input	$f(x, y)$ $V(x, y, z, t)$	$\overrightarrow{F}(x, y, z)$

We will mostly be here for the rest of the semester.

Note that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2 \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2$$

$$y'(x) = 3x^2 \qquad y'(t) = 3t^2 \qquad x'(t) = 3t^2$$

$$y' = 3x^2 \qquad y' = 3t^2 \qquad x' = 3t^2$$

are basically all the same ODE, just written using different letters or notations.

Note that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2 \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2$$

Function (output)

$$y'(x) = 3x^2 \qquad y'(t) = 3t^2 \qquad x'(t) = 3t^2$$

Variable (input)

$$y' = 3x^2 \qquad y' = 3t^2 \qquad x' = 3t^2$$

are basically all the same ODE, just written using different letters or notations.

However,

$$y' = 3y^2 \qquad x' = 3x^2$$

are *very different* from the first set (these two are equivalent to each other).

Vocabulary - order

The **order** of an ODE is the highest *derivative* that occurs in the equation.

- $\dfrac{dy}{dx} = 5x^{10}$ has order 1. We say this is a first-order ODE.
- $\dfrac{d^2y}{dx^2} = 0$ has order 2. We say this is a second-order ODE.
- $\dfrac{d^2y}{dx^2} + x^9 \cdot \dfrac{dy}{dx} = \sin(x)$ has order 2 also.

Any order ≥ 2 can be called higher-order.

We will see several different kinds of first-order ODEs in this class.
The second-order ODEs we study will usually be “linear” ODEs.

Task: Solve the ODE $y' = 8x^3$.

- This is the same as saying “Describe all $y = f(x)$ for which $y' = 8x^3$.”

Task: Solve the ODE $y' = 4y$.

- This is the same as “Describe all functions $y = f(x)$ for which $f'(x) = 4 \cdot f(x)$”, but that is not as easy as the previous example.

ODE vs. IVP

An **initial condition** (**IC**), also called an **initial value**, is a piece of information giving the value of a function or its derivative for a particular value of the input variable.

- Examples: $f(0) = 4$ or $x(0) = 8$ or $y'(0) = -2$ or $y(3) = 1$.

Note that, despite the word "initial", an IC does not have to be about $t = 0$.

An **initial value problem** (**IVP**), sometimes called a **Cauchy problem**, is a differential equation along with one or more initial values.

- Example: "$y' = 3x^2$ and $y(1) = 5$" is one IVP.
- Example: "$y' = 3x^2$ and $y(0) = 10$" is another IVP.

“Solve the IVP $y' = \sin(x),\ y(0) = 4$.”

means exactly the same as

$$\text{“If } \overbrace{\underbrace{y' = \sin(x)}_{\text{differential equation}} \text{ and } \underbrace{y(0) = 4}_{\text{initial value}}}^{\text{initial value problem}}\text{, find } y(x)\text{.”}$$

A **particular solution** to an IVP or ODE is a solution that does not contain arbitrary constants. (It might still contain constants from the ODE.)
For an IVP, this also means that the solution satisfies the initial conditions.

- Example: the particular solution to $y' = 12x^2$ and $y(2) = 12$ is $y = 4x^3 - 4$.
- Example: the particular solution to $y' = kx^2$ and $y(0) = 9$ is $y = \frac{k}{3}x^3 + 9$.

A **general solution** to an IVP or ODE ignores initial condition(s) and describes all possible solutions of an ODE. This requires using arbitrary constants. In this class we usually use C for one constant and C_1, C_2, ... for multiple constants.

- Example: the general solution to $y' = 3x^2$ and $y(1) = 5$ is $y' = x^3 + C$.
- Example: the general soln. to $y''(t) = -9.8$ is $y = y_0 + v_0 t + \frac{1}{2}(-9.8)t^2$.

Example: Solve the IVP $y'(x) = x^6 + e^{4x}, \quad y(0) = 3.$

First, find the general solution.

Then use the initial condition to figure out C.

Vocabulary so far

Differential equation (diff. eq.) – an equation with a derivative in it.
Partial differential equation (PDE) – for function with multiple inputs.
Ordinary differential equation (ODE) – for function with one input.

Initial condition – info about function or derivative at a specific input.
Initial value problem – a diff. eq. together with an initial condition.

First-order, second-order, etc. – highest derivative is y', y'', etc. (or x', x'', etc.).

Particular solution – does not have arbitrary constants (no "$+C$").
General solution – describes all possible solutions, ignoring any initial conditions.

Vocabulary so far

Differential equation (diff. eq. or DE)	równanie różniczkowe
Initial condition (IC) or initial value	warunek początkowy
Initial value problem (IVP)	zagadnienie początkowe
First order, etc.	pierwszego rzędu, itd.
Particular solution or specific solution	rozwiązanie specjalne
General solution	rozwiązanie ogólne

Types of ODEs

There are many words we can use to classify differential equations. (We will learn these definitions later.)

- first-order, second-order, etc.
- autonomous
- separable
- linear
- homogenous
- constant coefficients

Some of these categories can overlap. For example, we could have a

"homogeneous 2nd-order linear ODE with constant coefficients".

Direct ODEs, Autonomous ODEs

A **direct first-order ODE** is one where the derivative equals a function of the input variable *only*. As a formula,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x). \text{ or } \frac{dy}{dt} = f(t) \text{ or } x'(t) = f(t)$$

The explicit general solution to this is always found by integrating:

$$y = \int f(x)\,\mathrm{d}x.$$

Getting a nice formula for $\int f(x)\,\mathrm{d}x$ can be easy, difficult, or impossible, depending on f.

Some textbooks use the phrase "separable in x" or "directly integrable".

Direct ODEs, Autonomous ODEs

An **autonomous first-order ODE** for $y(x)$ can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(y).$$

Remember that we can use other letters. Therefore

- $y'(x) = y^3$
- $y'(t) = \frac{1}{y}$
- $\frac{\mathrm{d}x}{\mathrm{d}t} = rx - rx^2$
- $x' + x^2 = 0$

are all autonomous.

The particular solution to the IVP

$$y'(x) = y, \quad y(0) = 6$$

is…

A. $y(x) = 6e^x$

B. $y(x) = 6x^2$

C. $y(x) = x^2 + 6$

D. $y(x) = e^x + 6$

E. $y(x) = x^2 + e^x$

Solving autonomous ODEs

Example: $y' = \sqrt{y}$.

Solving autonomous IVPs

Example: $x' = e^{3x}, \quad x(0) = 4.$

Solving autonomous ODEs

Example: $y' = \dfrac{1}{1 + e^y}$.

An **explicit solution** (often just called a **solution**) to a differential equation is a function that satisfies the ODE, either everywhere on an interval.

- Example: One explicit solution to the ODE

$$y' = 3y^2$$

is $y = \frac{-1}{3x}$.

For today, do not worry about HOW we can get from $y' = 3y^2$ to $y = -1/(3x)$.

An **explicit solution** (often just called a **solution**) to a differential equation is a function that satisfies the ODE, either everywhere on an interval.

- Example: One explicit solution to the ODE

$$y' = 3y^2$$

is $y = \frac{-1}{3x}$. Another is $y = \frac{-1}{3x+10}$. We can also say $y = \frac{-1}{3x+C}$ is an explicit solution (it is also the “general solution”).

An **explicit solution** (often just called a **solution**) to a differential equation is a function that satisfies the ODE, either everywhere on an interval.

- Example: One explicit solution to the ODE

$$y' = 3y^2$$

is $y = \frac{-1}{3x}$. Another is $y = \frac{-1}{3x+10}$. We can also say $y = \frac{-1}{3x+C}$ is an explicit solution (it is the "general explicit solution").

An **implicit solution** is actually an *equation* that involves the output variable, satisfies the ODE, and does not include any derivatives.

- Example: $3xy = -1$ is an implicit solution to $y' = 3y^2$.
- Example: $3xy + Cy = -1$ is an implicit general solution to $y' = 3y^2$.

Vote: Which of the following is an explicit solution to $y' = 2xy$?

A. $y = e^{x^2}$

B. $y = x^2 y$

C. $y = x\sin(x)$

D. $y = \sin(x+y)$

E. None of the above

Survey: Which of the following is an explicit solution to $x''(t) = -4x$?

A. $\sin(x)$

B. $-2x^2$

C. $\sin(4t)$

D. $5\sin(2t)$

E. $-2\sin(x)^2$

F. None of the above

Vocabulary so far

Differential equation (DE)	równanie różniczkowe
Ordinary differential equation (ODE)	rów. róż. zwyczajne
First-order, second-order, etc.	pierwszego rzędu, itd.
Explicit solution or solution	rozwiązanie jawne
Implicit solution	rozwiązanie niejawne
General solution	rozwiązanie ogólne
Particular solution or specific solution	rozwiązanie specjalne
Initial condition (IC) or initial value	warunek początkowy
Initial value problem (IVP)	zagadnienie początkowe