Analysis 2

30 April 2024

Warm-up: $\int \frac{1}{y}\,\mathrm{d}y = ?$

Depending on context, this could be any of

- $\ln y + C$
- $\ln |y| + C$
- $\ln y + g(x)$
- $\ln |y| + g(x)$
- $\ln y$, kind of

Vocabulary so far

Last week

Differential equation (diff. eq.) – an equation with a derivative in it.
Ordinary differential equation (ODE) – for function with one input.
Partial differential equation (PDE) – for function with multiple inputs.

Initial condition – info about function or derivative at a specific input.
Initial value problem – a diff. eq. together with an initial condition.

First-order, second-order, etc. – highest derivative is y', y'', etc. (or x', x'', etc.).

Particular solution – does not have arbitrary constants (no "$+C$").
General solution – describes all possible solutions, ignoring any initial conditions.

Example: Solve the PDE $f'_x = \dfrac{9}{2x}, \quad f'_y = \dfrac{1}{y}$.

Method A:

$f'_x \to f + g(y) \to f'_y$

Match this with f'_y from task to get g'.

$g' \to g \to f$

or

Method B:

$f'_y \to f + g(x) \to f'_x$

Match this with f'_x from task to get g'.

$g' \to g \to f$

Either way, the answer is $\boxed{f = \frac{9}{2}\ln(x) + \ln(y) + C}$, which can also be written as $f = \ln(x^{9/2}y) + C$.

Example: Solve the IVP $y'(x) = x^6 + e^{4x}, \quad y(0) = 3.$

First, find the general solution.

Then use the initial condition to find C.

Answer: $y = \frac{1}{7}x^7 + \frac{1}{4}e^{4x} + \frac{11}{4}$

Example 2: Solve the IVP $y''(t) = 6t + 7, \quad y(0) = -8, \quad y'(1) = 12.$

First, find the general solution.

Then use the initial conditions to find constants.

Answer: $y = t^3 + \frac{7}{2}t^2 + 2t - 8$

Types of ODEs

There are many words we can use to classify differential equations. (We will learn these definitions later.)

- first-order, second-order, etc.
- directly integrable
- autonomous
- separable
- linear
- homogenous
- non-homogeneous
- constant coefficients

Some of these categories can overlap. For example, we could have a

"homogeneous 2nd-order linear ODE with constant coefficients".

Direct ODEs, Autonomous ODEs

A **direct first-order ODE** is one where the derivative equals a function of the input variable *only*. As a formula,

$$\frac{dy}{dx} = f(x). \quad \text{or} \quad \frac{dy}{dt} = f(t) \quad \text{or} \quad x'(t) = f(t)$$

The explicit general solution to this is always found by integrating:

$$y = \int f(x)\, dx.$$

Getting a nice formula for $\int f(x)\, dx$ can be easy, difficult, or impossible, depending on f.

Some textbooks use the phrase "separable in x" or "directly integrable".

Direct ODEs, Autonomous ODEs

An **autonomous** **first-order ODE** for $y(x)$ can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(y).$$

Remember that we can use other letters. Therefore

- $y'(x) = y^3$
- $y'(t) = \frac{1}{y}$
- $\frac{\mathrm{d}x}{\mathrm{d}t} = rx - rx^2$
- $x' + x^2 = 0$

are all autonomous.

The particular solution to the IVP

$$y'(x) = 2y, \quad y(0) = 6$$

is…

A. $y(x) = x^2 + 6$

B. $y(x) = y^2 + 6$

C. $y(x) = 6x^2$

D. $y(x) = 6e^{2x}$

E. $y(x) = e^{2x} + 5$

F. None of the above

You should be able to answer the previous multiple-choice question *without* knowing how to solve the autonomous ODE $y' = 2y$.

- This kind of task *can* be on Quiz 4.
- Also Quiz 4: solving direct ODEs/IVPs (like $y' = 2x$) and direct PDEs/IVPs.

Now we will see how to get from the ODE $y' = 2y$ to the function $y = Ce^{2x}$.

- The task "Solve $y' = 2y$" could be part of Quiz 5, but not Quiz 4.

Solving autonomous ODEs

Task: Solve $y' = 2y$.

$$\frac{dy}{dx} = 2y$$

$$\frac{1}{2y}\,dy = dx$$

$$\int \frac{1}{2y}\,dy = \int 1\,dx$$

implicit soln → $\frac{1}{2}\ln(y) = x + C$

...

explicit soln → $y = Ce^{2x}$

Comments:

- It may feel like cheating to split dy and dx apart, but it's helpful.
- We don't need $\ln(y) + C_1 = ax + C_2$ because we could always subtract C_1 from both sides to just have one constant.
- The C here is not the same C from before, but in both cases it means "any constant".

An **explicit solution** (often just called a **solution**) to a differential equation is a function that satisfies the ODE, either everywhere on an interval.

- Example: One explicit solution to the ODE

$$y' = 3y^2$$

is $y = \frac{-1}{3x}$.

An **explicit solution** (often just called a **solution**) to a differential equation is a function that satisfies the ODE, either everywhere on an interval.

- Example: One explicit solution to the ODE

$$y' = 3y^2$$

is $y = \frac{-1}{3x}$. Another is $y = \frac{-1}{3x+10}$. We can also say $y = \frac{-1}{3x+C}$ is an explicit solution (it is the "general explicit solution").

An **implicit solution** is actually an *equation* that involves the output variable, satisfies the ODE, and does not include any derivatives.

- Example: $3xy = -1$ is an implicit solution to $y' = 3y^2$.
- Example: $3xy + Cy = -1$ is an implicit general solution to $y' = 3y^2$.

Describe functions of y only that have $(?)' = \frac{1}{y}$.

- Answer: $\ln(y) + C$
- This could be part of solving an autonomous ODE.
  - Although it might be more helpful to just use $\ln(y)$ without the $+C$, as we saw in the previous example.

Describe functions of x and y that have $(?)'_y = \frac{1}{y}$.

- Answer: $\ln(y) + g(x)$
- This could be part of solving a PDE.

Solving autonomous ODEs

Example: $y' = \sqrt{y}$.

Answer: $y = (\frac{1}{2}x + C)^2$

Note that solving $x' = \sqrt{x}$ would be done exactly the same way, just with different letters.

...

$$x = (\frac{1}{2}t + C)^2$$

But $y' = \sqrt{x}$ is very different.

The step $\frac{1}{\sqrt{y}}\mathrm{d}y = \mathrm{d}x$ is called "separation of variables".

Solving autonomous ODEs

Task: $x' = e^{3x}$.

Implicit solution: $\frac{-1}{3}e^{-3x} = t + C$

Explicit solution: $x = \frac{-1}{3}\ln(C - 3t)$

The step $e^{-3x}\mathrm{d}x = \mathrm{d}t$ is called "separation of variables".

Schedule

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
29 April ✡️ Last Night of Passover	30 April (today)	1 May ✗ 🇪🇺 Labour Day	2 ✗	3 ✗ 🇵🇱 Constitution Day	4 ✗ Quiz 4 extra point	5 ✗
6 Quiz 4 extra pt	7 Lecture	8 Problem Session Quiz 4	9	10	11 ✗	12 ✗
13	14 Lecture	15 Problem Session	16	17 Quiz 5 extra point	18 ✗	19 ✗