Analysis 2, Summer 2024

List 1

Calculations with multi-variable functions

37. State whether each is a "scalar" or "vector":
    (a) temperature scalar
    (b) position vector
    (c) voltage scalar
    (d) electric field vector
    (e) time scalar
    (f) force vector
    (g) height scalar

38. Re-write $\begin{cases} x = \cos(t) \\ y = t^2 \end{cases}$ as a single equation using vectors. Any of these formats are okay: $\vec{r}(t) = \cos(t)\hat{\imath} + t^2\hat{\jmath}$ or $\vec{r} = \cos t\,\hat{\imath} + t^2\hat{\jmath}$ or $\vec{r}(t) = \begin{bmatrix} \cos t \\ t^2 \end{bmatrix}$ or $\vec{r} = \begin{bmatrix} \cos t \\ t^2 \end{bmatrix}$

39. If $\vec{r} = 9\hat{\jmath} - \hat{k}$ describes a point in 3D space, what is the z-coordinate? -1

40. More **Analysis 1 review**: Calculate...
    (a) $(e^{5t})' = 5e^{5t}$
    (b) $\big(\ln(8t)\big)' = \dfrac{1}{t}$
    (c) $\frac{\mathrm{d}}{\mathrm{d}t}\left[\sqrt{t^6 + \sin(\pi t)}\,\right] = \dfrac{6t^5 + \cos(\pi t)\cdot\pi}{2\sqrt{\sin(\pi t)}}$
    (d) $\displaystyle\int 2t^7\sqrt{1+t^8}\,\mathrm{d}t = \tfrac{1}{6}(1+t^8)^{3/2} + C$
    (e) $\displaystyle\int_0^1 2t^7\sqrt{1+t^8}\,\mathrm{d}t = \dfrac{2\sqrt{2}-1}{6}$
    (f) $\displaystyle\int_0^{\pi/4} \cos(t)\cos(\sin(t))\,\mathrm{d}t = \sin(\tfrac{1}{\sqrt{2}})$

41. For the vector function $\vec{r}(t) = e^{5t}\hat{\imath} + \ln(8t)\hat{\jmath}$, calculate
    (a) $|\vec{r}|$, also written $|\vec{r}(t)| = \sqrt{e^{10t} + (\ln(8t))^2}$
    (b) $\vec{r}\,' = \vec{r}\,'(t) = 5e^{5t}\hat{\imath} + \dfrac{1}{t}\hat{\jmath}$
    (c) $|\vec{r}\,'| = \sqrt{25e^{10t} + \dfrac{1}{t^2}}$
    (d) $|\vec{r}|' = \dfrac{10e^{10t} + 2\ln(8t)\frac{1}{t}}{2\sqrt{e^{10t} + (\ln(8t))^2}}$

42. Calculate both $|\vec{r}\,'|$ and $|\vec{r}|'$ for $\vec{r} = \begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix}$.

$|\vec{r}\,'| = \big|[-3\sin 3t, 3\cos 3t]\big| = \sqrt{(-3\sin 3t)^2 + (3\cos 3t)^2} = \sqrt{9(c^2+s^2)} = \boxed{3}$.
$|\vec{r}|' = \left(\sqrt{(\cos 3t)^2 + (\sin 3t)^2}\right)' = \left(\sqrt{1}\right)' = \boxed{0}$

43. If $f(x,y,z) = 7xy^3\sin(x+z)$ and $x = t^2$ and $y = e^t$ and $z = t^3$, write a formula for $f(\vec{r}(t)) = f(x(t), y(t), z(t))$ using t as the only variable. $\boxed{f = 7t^2e^{3t}\sin(t^2+t^3)}$

The **path integral** of $f : \mathbb{R}^n \to \mathbb{R}$ along the curve C traced by $\vec{r} : [a,b] \to \mathbb{R}^n$ is

$$\int_C f\,\mathrm{d}s \quad = \quad \int_a^b f(\vec{r}(t))\big|\vec{r}\,'(t)\big|\mathrm{d}t.$$

44. Calculate $\int_a^b f(\vec{r}(t))\big|\vec{r}\,'(t)\big|\mathrm{d}t$ for the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = x^3 + y^3$$

and the curve $\vec{r} : [0,4] \to \mathbb{R}^2$ given by

$$\vec{r}(t) \quad = \quad x(t)\hat{\imath} + y(t)\hat{\jmath} \quad = \quad 2t\hat{\imath} - t\hat{\jmath}.$$

$$\vec{r}\,' = 2\hat{\imath} - \hat{\jmath}$$
$$|\vec{r}\,'| = \sqrt{(2)^2 + (-1)^2} = \sqrt{5}$$
$$f(\vec{r}(t)) = (2t)^3 + (-t)^3 = 7t^3$$
$$f(\vec{r}(t))|\vec{r}\,'| = 7\sqrt{5}t^3$$
$$\int_C f\,\mathrm{d}s = \int_0^4 7\sqrt{5}t^3\,\mathrm{d}t = \frac{7\sqrt{5}t^4}{4}\Bigg|_{t=0}^{t=4} = \boxed{448\sqrt{5}}$$

45. Integrate

$$f(x,y) = \frac{x^4}{y}$$

over the curve parameterized by

$$\vec{r}(t) = t^2\hat{\imath} + t^{-2}\hat{\jmath}, \quad 0 \le t \le 1.$$

$\boxed{\dfrac{2\sqrt{2}-1}{6}}$ (See Task 40(e))

46. Integrate

$$f(x,y,z) = \frac{\ln(x)e^z}{\sqrt{1+y^2+y^2e^{2y}}}$$

over the curve parameterized by

$$\vec{r}(t) = e^t\hat{\imath} + t\hat{\jmath} + \ln(t)\hat{k}, \qquad 1 \le t \le \sqrt{23}.$$

$$\vec{r}' = e^t\hat{\imath} + \hat{\jmath} + \tfrac{1}{t}\hat{k}$$

$$|\vec{r}'| = \sqrt{e^{2t} + 1 + t^{-2}}$$

$$f(\vec{r}(t)) = \frac{\ln(e^t)e^{\ln(t)}}{\sqrt{1 + t^2 + t^2e^{2t}}} = \frac{t \cdot t}{\sqrt{t^2(t^{-2} + 1 + e^{2t})}}$$

$$f(\vec{r}(t))|\vec{r}'| = \frac{t \cdot t}{\sqrt{t^2(t^{-2} + 1 + e^{2t})}} \cdot \sqrt{e^{2t} + 1 + t^{-2}} = t$$

$$\int_C f \, \mathrm{d}s = \int_1^{\sqrt{23}} t \, \mathrm{d}t = \frac{t^2}{2}\bigg|_{t=1}^{t=\sqrt{23}} = \boxed{11}$$

47. Integrate $x \cos y$ over the curve $\vec{r} = [5, \sin t]$ with $0 \le t \le \pi/4$. $\boxed{5\sin(\frac{1}{\sqrt{2}})}$ (See Task 40(f), multiplied by 5)

> The **partial derivative of $f(x, y)$ with respect to x** can be written as any of
>
> $$f_x'(x, y) \qquad f_x' \qquad D_x f(x, y) \qquad D_x f \qquad \partial_x f \qquad \frac{\partial f}{\partial x}.$$
>
> Officially, it is defined as $\lim_{h\to 0} \dfrac{f(x+h, y) - f(x, y)}{h}$, but in practice it is calculated by thinking of every letter other than x as a constant.
>
> Similarly, the partial derivative of f with respect to any one variable involves thinking of every other variable as constant.

48. Give the partial derivative of

$$f(x, y) = xy^3 + x^2 \sin(xy) - 2^x$$

with respect to x, which is a new function with two inputs. We can write $f_x'(x, y)$ or f_x' or $\dfrac{\partial f}{\partial x}$ or $\dfrac{\partial}{\partial x} f$ or $\dfrac{\partial}{\partial x}\left[xy^3 + x^2 \sin(xy) - 2^x\right]$ for this function.

It may help to think about $\frac{\mathrm{d}}{\mathrm{d}x}\left[ax + x^2 \sin(bx) - 2^x\right]$, where a, b, c are constants.

$\boxed{y^3 + x^2 y \cos(xy) + 2x \sin(xy) - 2^x \ln(2)}$

49. Give the partial derivative of

$$f(x, y) = xy^3 + x^2 \sin(xy) - 2^x$$

with respect to y, which is a new function with two inputs. We can write $f_y'(x, y)$ or f_y' or $\dfrac{\partial f}{\partial y}$ or $\dfrac{\partial}{\partial y} f$ or $\dfrac{\partial}{\partial y}\left[xy^3 + x^2 \sin(xy) - 2^x\right]$ for this function.

It may help to think about $\frac{\mathrm{d}}{\mathrm{d}t}\left[at^3 + b\sin(ct) - d\right]$, where a, b, c, d are constants.

$\boxed{3xy^2 + x^3 \cos(xy)}$

50. Find the functions $\frac{\partial}{\partial x}[y^x]$ and $\frac{\partial}{\partial y}[y^x]$.

$\boxed{f_x' = y^x \ln(y), f_y' = xy^{x-1}}$

51. Calculate the partial derivative of $f(x,y) = y^x$ with respect to x at the point $(5,2)$, which is a number. We can write $f'_x(5,2)$ or $\frac{\partial f}{\partial x}(5,2)$ or $\left.\frac{\partial f}{\partial x}\right|_{\substack{x=5\\y=2}}$ for this. $\boxed{\ln(9) \approx 2.19722}$

52. Calculate the partial derivative of $f(x,y) = y^x$ with respect to y at the point $(5,2)$, which is a number. We can write $f'_y(5,2)$ or $\frac{\partial f}{\partial y}(5,2)$ or $\left.\frac{\partial f}{\partial y}\right|_{\substack{x=5\\y=2}}$ for this.

$\boxed{5 \cdot 2^4 = 80}$

53. Calculate f'_x and f'_y and f'_z for $f(x,y,z) = \dfrac{y}{x^3+z}$.

$$\boxed{f'_x = \frac{-3x^2y}{(x^3+z)^2}, \quad f'_y = \frac{1}{x^3+z}, \quad f'_z = \frac{-y}{(x^3+z)^2}}$$

54. Find each of the following partial derivatives:

(a) $\dfrac{\partial}{\partial x}\left[x^2y\right] = \boxed{2xy}$

(b) $\dfrac{\partial}{\partial y}\left[x^2y\right] = \boxed{x^2}$

(c) $\dfrac{\partial}{\partial x}\left[xyz\right] = \boxed{yz}$

(d) $\dfrac{\partial}{\partial x}\left[x^y\right] = \boxed{yx^{y-1}}$

(e) $\dfrac{\partial}{\partial y}\left[x^y\right] = \boxed{x^y\ln(x)}$

(f) $\dfrac{\partial}{\partial r}\left[\pi r^2h\right] = \boxed{2\pi rh}$

(g) $\dfrac{\partial}{\partial z}\left[xyz\right] = \boxed{xy}$

(h) $\dfrac{\partial}{\partial z}\left[e^{xyz}\right] = \boxed{e^{xyz}xy}$

(i) $\dfrac{\partial}{\partial a}\left[(a^2+b^2)\right] = \boxed{2a}$

(j) $\dfrac{\partial}{\partial y}\left[x^2\sin(xy)\right] = \boxed{x^3\cos(xy)}$

(k) $\dfrac{\partial}{\partial y}\left[\ln(5x)\right] = \boxed{0}$

(ℓ) $\dfrac{\partial}{\partial y}\left[\dfrac{\cos(x+y)}{2x+5y}\right] = \boxed{\dfrac{-(2x+5y)\sin(x+y) - 5\cos(x+y)}{(2x+5y)^2}}$

55. Calculate u'_x, u'_y, v'_x, and v'_y for the functions $u(x,y) = \dfrac{x^2}{y}$ and $v(x,y) = x - y^2$.

$$\boxed{u'_x = 2xy^{-1}, \quad u'_y = -x^2y^{-2}, \quad v'_x = 1, \quad v'_y = -2y}$$

For a function $f(x,y)$, the **second derivative with respect to x twice** is

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

and can be written as $\dfrac{\partial^2 f}{\partial x^2}$ or as f''_{xx}.

Similarly, the **second d. with respect to y twice** is $f''_{yy} = \dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)$.

The **mixed partial derivatives** are

$$f''_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \qquad \text{and} \qquad f''_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

56. Calculate f''_{xx} for $f = e^{xy}$ by calculating f'_x and then $\frac{\partial}{\partial x}(f'_x)$.

$f'_x = ye^{xy}$, and $\boxed{f''_{xx} = y^2 e^{xy}}$

57. Calculate f''_{yy} for $f = y^x$ by calculating f'_y and then $\frac{\partial}{\partial y}(f'_y)$.

$f'_y = xy^{x-1}$, and $\boxed{f''_{yy} = x^2 y^{x-2}}$

58. For $f = \dfrac{x}{y}$,

(a) Calculate f''_{xy} by calculating f'_x and then $\frac{\partial}{\partial y}(f'_x)$. $f'_x = \dfrac{1}{y}$ and $\boxed{f''_{xy} = -\dfrac{1}{y^2}}$

(b) Calculate f''_{yx} by calculating f'_y and then $\frac{\partial}{\partial x}(f'_y)$. $f'_y = -\dfrac{x}{y^2}$ and $\boxed{f''_{yx} = -\dfrac{1}{y^2}}$

59. For $g = e^{\cos(x)} + \ln(y^3)$,

(a) Calculate g''_{xy} by calculating g'_x and then $\frac{\partial}{\partial y}(g'_x)$.

$f'_x = \sin(x)\left(-e^{\cos(x)}\right)$ and $\boxed{f''_{xy} = 0}$

(b) Calculate g''_{yx} by calculating g'_y and then $\frac{\partial}{\partial x}(g'_y)$.

$f'_y = \frac{3y^2}{y^3} = \frac{3}{y}$ and $\boxed{f''_{yx} = 0}$

☆60. Give an example of a function $f(x, y)$ for which $f'_x = y^4$ and $f'_y = x^4$, or explain why no such $f(x, y)$ exists.

Such an f $\boxed{\text{does not exist}}$. If it did, then $f''_{xy} = \frac{\partial}{\partial y}(f'_x)$ would be $\frac{\partial}{\partial y}(y^4) = 4y^3$, and $f''_{yx} = \frac{\partial}{\partial x}(f'_y)$ would be $\frac{\partial}{\partial x}(x^4) = 4x^3$. But f''_{xy} and f''_{yx} must be equal.

61. Give all the second partial derivatives of $f(x, y) = x\ln(xy)$.

$$\boxed{f''_{xx} = \frac{1}{x}, \quad f''_{xy} = f''_{yx} = \frac{1}{y}, \quad f''_{yy} = \frac{-x}{y^2}}$$