Analysis 2, Summer 2024

List 1

Calculations with multi-variable functions

37. State whether each is a "scalar" or "vector":

    (a) temperature

    (b) position

    (c) voltage

    (d) electric field

    (e) time

    (f) force

    (g) height

38. Re-write $\begin{cases} x = \cos(t) \\ y = t^2 \end{cases}$ as a single equation using vectors.

39. If $\vec{r} = 9\hat{\jmath} - \hat{k}$ describes a point in 3D space, what is the z-coordinate?

40. More **Analysis 1 review**: Calculate...

    (a) $(e^{5t})'$

    (b) $\big(\ln(8t)\big)'$

    (c) $\frac{\mathrm{d}}{\mathrm{d}t}\left[\sqrt{t^6 + \sin(\pi t)}\,\right]$

    (d) $\displaystyle\int 2t^7\sqrt{1+t^8}\,\mathrm{d}t$

    (e) $\displaystyle\int_0^1 2t^7\sqrt{1+t^8}\,\mathrm{d}t$

    (f) $\displaystyle\int_0^{\pi/4} \cos(t)\cos(\sin(t))\,\mathrm{d}t$

    Simplify your answer for (b).

41. For the vector function $\vec{r}(t) = e^{5t}\hat{\imath} + \ln(8t)\hat{\jmath}$, calculate

    (a) $|\vec{r}|$, also written $|\vec{r}(t)|$ (b) $\vec{r}\,' = \vec{r}\,'(t)$ (c) $|\vec{r}\,'|$ (d) $|\vec{r}|'$

42. Calculate both $|\vec{r}\,'|$ and $|\vec{r}|'$ for $\vec{r} = \begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix}$.

43. If $f(x,y,z) = 7xy^3\sin(x+z)$ and $x = t^2$ and $y = e^t$ and $z = t^3$, write a formula for $f(\vec{r}(t)) = f(x(t),y(t),z(t))$ using t as the only variable.

The **path integral** of $f : \mathbb{R}^n \to \mathbb{R}$ along the curve C traced by $\vec{r} : [a,b] \to \mathbb{R}^n$ is

$$\int_C f\,\mathrm{d}s \;=\; \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,\mathrm{d}t.$$

44. Calculate $\displaystyle\int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,\mathrm{d}t$ for the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

    $$f(x,y) = x^3 + y^3$$

    and the curve $\vec{r} : [0,4] \to \mathbb{R}^2$ given by

    $$\vec{r}(t) \;=\; x(t)\hat{\imath} + y(t)\hat{\jmath} \;=\; 2t\hat{\imath} - t\hat{\jmath}.$$

45. Integrate

$$f(x,y) = \frac{x^4}{y}$$

over the curve parameterized by

$$\vec{r}(t) = t^2\hat{\imath} + t^{-2}\hat{\jmath}, \quad 0 \le t \le 1.$$

46. Integrate

$$f(x,y,z) = \frac{\ln(x)e^z}{\sqrt{1+y^2+y^2e^{2y}}}$$

over the curve parameterized by

$$\vec{r}(t) = e^t\hat{\imath} + t\hat{\jmath} + \ln(t)\hat{k}, \qquad 1 \le t \le \sqrt{23}.$$

47. Integrate $x\cos y$ over the curve $\vec{r} = [5, \sin t]$ with $0 \le t \le \pi/4$.

The **partial derivative of $f(x,y)$ with respect to x** can be written as any of

$$f'_x(x,y) \qquad f'_x \qquad D_x f(x,y) \qquad D_x f \qquad \partial_x f \qquad \frac{\partial f}{\partial x}.$$

Officially, it is defined as $\lim\limits_{h\to 0} \dfrac{f(x+h,y) - f(x,y)}{h}$, but in practice it is calculated by thinking of every letter other than x as a constant.

Similarly, the partial derivative of f with respect to any one variable involves thinking of every other variable as constant.

48. Give the partial derivative of

$$f(x,y) = xy^3 + x^2\sin(xy) - 2^x$$

with respect to x, which is a new function with two inputs. We can write $f'_x(x,y)$ or f'_x or $\dfrac{\partial f}{\partial x}$ or $\dfrac{\partial}{\partial x}f$ or $\dfrac{\partial}{\partial x}\big[xy^3 + x^2\sin(xy) - 2^x\big]$ for this function.

It may help to think about $\frac{\mathrm{d}}{\mathrm{d}x}\big[ax + x^2\sin(bx) - 2^x\big]$, where a, b, c are constants.

49. Give the partial derivative of

$$f(x,y) = xy^3 + x^2\sin(xy) - 2^x$$

with respect to y, which is a new function with two inputs. We can write $f'_y(x,y)$ or f'_y or $\dfrac{\partial f}{\partial y}$ or $\dfrac{\partial}{\partial y}f$ or $\dfrac{\partial}{\partial y}\big[xy^3 + x^2\sin(xy) - 2^x\big]$ for this function.

It may help to think about $\frac{\mathrm{d}}{\mathrm{d}t}\big[at^3 + b\sin(ct) - d\big]$, where a, b, c, d are constants.

50. Find the functions $\frac{\partial}{\partial x}\big[y^x\big]$ and $\frac{\partial}{\partial y}\big[y^x\big]$.

51. Calculate the partial derivative of $f(x,y) = y^x$ with respect to x at the point $(5,2)$, which is a number. We can write $f'_x(5,2)$ or $\frac{\partial f}{\partial x}(5,2)$ or $\frac{\partial f}{\partial x}\Big|_{\substack{x=5\\y=2}}$ for this.

52. Calculate the partial derivative of $f(x,y) = y^x$ with respect to y at the point $(5,2)$, which is a number. We can write $f'_y(5,2)$ or $\frac{\partial f}{\partial y}(5,2)$ or $\left.\frac{\partial f}{\partial y}\right|_{\substack{x=5\\y=2}}$ for this.

53. Calculate f'_x and f'_y and f'_z for $f(x,y,z) = \dfrac{y}{x^3+z}$.

54. Find each of the following partial derivatives:

(a) $\dfrac{\partial}{\partial x}\left[x^2 y\right]$ (d) $\dfrac{\partial}{\partial x}\left[x^y\right]$ (g) $\dfrac{\partial}{\partial z}\left[xyz\right]$ (j) $\dfrac{\partial}{\partial y}\left[x^2\sin(xy)\right]$

(b) $\dfrac{\partial}{\partial y}\left[x^2 y\right]$ (e) $\dfrac{\partial}{\partial y}\left[x^y\right]$ (h) $\dfrac{\partial}{\partial z}\left[e^{xyz}\right]$ (k) $\dfrac{\partial}{\partial y}\left[\ln(5x)\right]$

(c) $\dfrac{\partial}{\partial x}\left[xyz\right]$ (f) $\dfrac{\partial}{\partial r}\left[\pi r^2 h\right]$ (i) $\dfrac{\partial}{\partial a}\left[(a^2+b^2)\right]$ (ℓ) $\dfrac{\partial}{\partial y}\left[\dfrac{\cos(x+y)}{2x+5y}\right]$

55. Calculate u'_x, u'_y, v'_x and v'_y for the functions $u(x,y) = \dfrac{x^2}{y}$ and $v(x,y) = x - y^2$.

For a function $f(x,y)$, the **second derivative with respect to x twice** is

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

and can be written as $\dfrac{\partial^2 f}{\partial x^2}$ or as f''_{xx}.

Similarly, the **second d. with respect to y twice** is $f''_{yy} = \dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)$.

The **mixed partial derivatives** are

$$f''_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \quad \text{and} \quad f''_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

56. Calculate f''_{xx} for $f = e^{xy}$ by calculating f'_x and then $\frac{\partial}{\partial x}(f'_x)$.

57. Calculate f''_{yy} for $f = y^x$ by calculating f'_y and then $\frac{\partial}{\partial y}(f'_y)$.

58. For $f = \dfrac{x}{y}$,

(a) Calculate f''_{xy} by calculating f'_x and then $\frac{\partial}{\partial y}(f'_x)$.

(b) Calculate f''_{yx} by calculating f'_y and then $\frac{\partial}{\partial x}(f'_y)$.

59. For $g = e^{\cos(x)} + \ln(y^3)$,

(a) Calculate g''_{xy} by calculating g'_x and then $\frac{\partial}{\partial y}(g'_x)$.

(b) Calculate g''_{yx} by calculating g'_y and then $\frac{\partial}{\partial x}(g'_y)$.

60. Give an example of a function $f(x,y)$ for which $f'_x = y^4$ and $f'_y = x^4$, or explain why no such $f(x,y)$ exists.

61. Give all the second partial derivatives of $f(x,y) = x\ln(xy)$.