Analysis 2, Summer 2024

List 10

Systems of linear ODEs / Review for Exam 2

A **system of two first-order linear ODEs** for $x(t)$ and $y(t)$ can be written as
$$\begin{cases} x' = a_{11}x + a_{12}y + b_1 \\ y' = a_{21}x + a_{22}y + b_2 \end{cases}$$
In this course, the systems will always be homogeneous (meaning $b_1 = b_2 = 0$) and the coefficients a_{ij} will always be constants.

241. (a) Re-write the system
$$\begin{cases} x' = 3y \\ y' = x + 2y \end{cases}$$
as a single second-order linear ODE for $x(t)$. Simplify to remove fractions.

(b) Re-write the system as a single ODE for $y(t)$.

(c) Find the general solution to the system.

242. Solve $\begin{cases} x' = 12x + 2y \\ y' = -12x + 23y. \end{cases}$

243. Solve $\begin{cases} x' = 3x - 2y \\ y' = 4x + 7y, \end{cases} \qquad x(0) = 8, \quad y(0) = -8.$

☆244. Solve $\begin{cases} x' = x - 2y - z \\ y' = -x + y + z \\ z' = x - z. \end{cases}$

245. Solve $\begin{cases} x' = -2x + y, & x(0) = 1 \\ y' = 3x - 2y, & y(0) = 9\sqrt{3} \end{cases}$

246. Complete the following table:

	Separable?	Homogeneous linear?	Non-homogeneous linear?
$y' = \sin(y)$	Yes	No	No
$y' = \sin(x)$			
$x' = \sin(t)$			
$x' = \sin(x)$			
$y' = y \cdot t$			
$y' = y + t$			
$y' = y$			
$y' = t$			

247. Is the ordinary differential equation $y' = x^3$

(a) first-order?
(b) second-order?
(c) third-order?
(d) direct?
(e) autonomous?
(f) separable?
(g) linear?
(h) linear and homogenous?
(i) linear with constant coefficients?

248. Solve the direct ODE $f' = 3x\cos(3x) + \sin(3x)$ for $f(x)$.

249. Solve the direct PDE $f'_x = \sin(xy) + xy\cos(xy)$, $f'_y = x^2\cos(xy) + 2y$ for $f(x, y)$.

250. Does the partial differential equation $f'_x = x^2y, f'_y = x^3y$ have a solution? Why or why not?

251. Solve the following PDEs, if solutions exist:

(a) $f'_x = x, \quad f'_y = y$

(b) $f'_x = x, \quad f'_y = x$

(c) $f'_x = y, \quad f'_y = x$

(d) $f'_x = x^2y^2, \quad f'_y = \frac{2}{3}x^3y$

(e) $f'_x = 3x^2 + y, \quad f'_y = 6x + 5y^4$

(f) $f'_x = 3x^2 + y, \quad f'_y = x + 5y^4$

(g) $f'_x = 6xy\, e^{x^2y}, \quad f'_y = 3x^2e^{x^2y}$

(h) $f'_x = \dfrac{\sin(y)}{x}, \quad f'_y = \cos(y)\ln(x)$

(i) $f'_x = x^2 + \dfrac{1}{y}, \quad f'_y = x^3 - \dfrac{x}{y^2}$

252. The general solution to the ODE

$$y' + \frac{y}{x} = x^2y^2$$

is $y = \dfrac{2}{Cx - x^3}$. Knowing this, give the particular solution to the IVP

$$y' + \frac{y}{x} = x^2y^2, \qquad y(1) = 3.$$

253. Is $y = x^3y^3$ a general solution or particular solution? Is it an implicit solution or an explicit solution?

254. Give the order of the differential equation

$$x^{11}y'' + e^xy = \sin(e^x).$$

(Do *not* try to solve this ODE.)

255. Which of the following ODEs is third-order?
(A) $x' = x^3$ (B) $x''' = \sin(x)$ (C) $y' + y'' = 3$ (D) $y'' = t^3 - 4t^2 + 1$

256. The general solution to the ODE

$$x' + x = \frac{2t}{x}$$

is $x = \pm\sqrt{Ce^{-2t} + 2t - 1}$. Knowing this, give the particular solution to the IVP

$$x' + x = \frac{2t}{x}, \qquad x(0) = -4.$$

257. Give an example of a first-order ODE for $y = y(t)$ in each of the following categories:

(a) direct, (b) autonomous,

(c) separable but not direct or autonomous,

(d) homogenous linear with constant coefficients,

(e) homogenous linear with non-constant coefficients,

(f) non-homogenous linear with constant coefficients,

(g) non-homogenous linear with non-constant coefficients.

258. Which category/categories from Task 257 does $x^2y' + y - 5 = 0$ belong to?

259. Give an example of a homogenous second-order linear ordinary differential equation with *non*-constant coefficients.

260. Re-write each of the following linear ODEs into the form $y' + a(x)y = f(x)$ or $y' + a(t)y = f(t)$ or $x' + a(t)x = f(t)$.

(a) $x^2y' = 7 - y$ (b) $x' = 12x$ (c) $y' = 9xy$ (d) $y' + t = -y$

261. Solve the following first-order ODEs:

(a) $y' = e^{(x^5)}x^4$

(b) $y' = y^2$

(c) $y' = x^3y^2$

(d) $x' - e^t x = 5e^t$

(e) $tx' + 3x = t^3$

(f) $y' = \sin(x) \cdot e^y$

(g) $x' = \sin(t) \cdot e^x$

(h) $y' = \sin(x) \cdot e^x$

(i) $y' + 3t^2y = 10t^2$

(j) $y' + 2y = \sin(t)$

(k) $y' - 5y^2 = xy^2$

(ℓ) $x \cdot x' = t^3$

262. Give two different particular solutions to $y' = xy$.

263. Give the Laplace transforms of

$$f(t) + g(t) \qquad \text{and} \qquad f(t)g(t)$$

for $f(t) = 3e^{2t}$ and $g(t) = \sin(5t)$.

264. Give the inverse Laplace transform of

$$F(s) = \frac{8 - 5s}{s^2 + 4}.$$

That is, what function $f(t)$ has $F(s) = \mathscr{L}\big[f(t)\big]$?

265. Give the Laplace transform of

$$y' + 6y + 5t$$

if $y(0) = 2$. Your answer should include s and also $Y = \mathscr{L}\big[y\big]$.

266. For the IVP $y' = y^3, \;\; y(0) = 5$, give

(a) an implicit general solution,

(b) an implicit particular solution,

(c) an explicit general solution,

(d) the explicit particular solution.

267. For the ODE $y' = 2xy$, give

(a) an implicit general solution,

(b) an implicit particular solution,

(c) the explicit general solution,

(d) an explicit particular solution.

268. Solve the ODE $y' + 6ty = e^{-3t^2}$.

269. Solve the IVP

$$y' + \cos(x)y = 12\cos(x), \qquad y(0) = 5.$$

270. Give the particular solution to $y' - 7y = e^{8t}$ with each of the following initial conditions: (a) $y(0) = 0$. (b) $y(0) = 4$. (c) $y(0) = 1$. (d) $y(1) = 0$

271. One of the following ODEs is autonomous. Solve that one and ignore the other.

$$x' + tx = 12 \qquad \text{or} \qquad y' + 3y = 12.$$

272. One of the following ODEs is separable. Solve that one and ignore the other.

$$x' + t^3 = 5t \qquad \text{or} \qquad y' + ty = 5.$$

273. One of the following ODEs is linear. Solve that one and ignore the other.

$$x' + 5t^4x = t^4 \qquad \text{or} \qquad y' + 5t^4y = y^4.$$

274. The second-order ODE

$$y'' + y' \cdot y = 0$$

has general solution

$$y(t) = C_1 + \frac{2C_1}{C_2e^{C_1t} - 1}.$$

Using this, give the solution to the IVP

$$y'' + y' \cdot y = 0, \quad y(0) = 0, \quad y'(0) = 8.$$

275. True or false:
   (a) Every first-order direct ODE is linear.
   (b) Every first-order autonomous ODE is linear.
   (c) Every first-order linear ODE is separable.
   (d) Every homogeneous first-order linear ODE is separable.
   (e) Every homogeneous first-order linear ODE is autonomous.
   (f) Every separable ODE is either direct or autonomous.
   (g) Every first-order linear ODE is either homogeneous or separable.

276. Complete the following table, where "CC" means constant coefficients and "hom." means homogeneous. *Note that each row must have exactly one checkmark.*

		CC hom. linear	CC non-hom. linear	non-CC hom. linear	non-CC non-hom. linear	not linear
	$y'' + 4y' = \sin(x)$	-	✔	-	-	-
	$x'' - t^2x = 0$	-	-	✔	-	-
	$x'' - tx^2 = 0$	-	-	-	-	✔
(a)	$y'' = t^2$					
(b)	$x'' = x^2$					
(c)	$x'' = t^2x$					
(d)	$y'' = x^2$					
(e)	$y'' + y' = e^t$					
(f)	$y'' + y' = y$					

277. Solve the following IVPs:

(a) $\frac{1}{2}y'' - 4y' + \frac{41}{2}y = 0, \quad y(0) = 1, \;\; y'(0) = -2.$

(b) $x'' + 7x' + 10x = 0, \quad x(0) = 13, \;\; x'(0) = 100.$

(c) $4x' - 3x = 10\cos(t), \quad x(0) = 1.$

(d) $t\,x' + 3x = t^3, \quad x(1) = \frac{1}{2}.$

(e) $x'' - x' - 2x = 0, \quad x(0) = 5, \;\; x'(0) = 8.$

(f) $x'' - x' + 2x = 0, \quad x(0) = 5, \;\; x'(0) = 8.$

(g) $y'' - 12y' + 36y = 6t^2 - 1, \quad y(0) = \frac{-1}{3}, \;\; y'(0) = \frac{1}{9}.$

(h) $\begin{cases} x' = x + y, & x(0) = 5 \\ y' = 2x, & y(0) = 3 \end{cases}$

278. One of the ODEs below is from a category that we have learned how to solve.[1] Solve that one and ignore the other two.

$$x' = x^7 \qquad \text{or} \qquad x' = t + x^7 \qquad \text{or} \qquad x'' = x^7$$

279. One of the ODEs below is from a category that we have learned how to solve.[1] Solve that one and ignore the other two.

$$x'' = tx + 7 \qquad \text{or} \qquad x'' = x + 7t \qquad \text{or} \qquad x'' = 7^x + t$$

280. One of the ODEs below is from a category that we have learned how to solve.[1] Solve that one and ignore the other two.

$$y''' + y = 1 \qquad \text{or} \qquad y'' + y^2 = 1 \qquad \text{or} \qquad y''' \cdot y = 1$$

[1] For first-order, these include directly, autonomous, separable, and linear. For higher-order, this is only linear with constant coefficients. For systems, this is only homogeneous linear systems with constant coefficients.