Analysis 2, Summer 2024

List 3

Directional derivatives, critical points

86. (a) Give the derivative of $\dfrac{x^3}{\sin(\pi y)}$ at the point $(4, \frac{1}{4})$ in the direction $\hat{\imath} = [1, 0]$.

    $f'_x(4, \frac{1}{4}) = \boxed{48\sqrt{2}}$

    (b) Give the derivative of $\dfrac{x^3}{\sin(\pi y)}$ at the point $(4, \frac{1}{4})$ in the direction $\hat{\jmath} = [0, 1]$.

    $f'_y(4, \frac{1}{4}) = \boxed{-64\sqrt{2}\pi}$

The **directional derivative** of $f(x, y)$ at the point (a, b) in the direction of the **unit** vector $\hat{u}$ (a vector of length 1) is written as $f'_{\hat{u}}(a, b)$ and can be calculated as

$$f'_{\hat{u}}(a, b) = \nabla f(a, b) \cdot \hat{u}.$$

87. For $f(x, y) = x^2 \sin(y)$, calculate the directional derivative at $(4, \frac{\pi}{3})$ in the direction $\hat{u} = \left[\frac{\sqrt{3}}{2}, \frac{1}{2}\right]$.

    $f'_{\hat{u}}(4, \frac{\pi}{3}) = \left[\frac{\sqrt{3}}{2}, \frac{1}{2}\right] \cdot [4\sqrt{3}, 8] = \frac{\sqrt{3}}{2}(4\sqrt{3}) + \frac{1}{2}(8) = \boxed{10}$.

88. What is the derivative of $f(x, y) = xe^y$ at the point $(3, 0)$ in the direction $[1, 1]$?

    $\nabla f = [e^y, xe^y]$ at $(x, y) = (3, 0)$ is $\nabla f = [e^0, 3e^0] = [1, 3]$, and the unit vector in the direction of $[1, 1]$ is

$$\hat{u} = \frac{[1, 1]}{\big|[1, 1]\big|} = \frac{[1, 1]}{\sqrt{1^2 + 1^2}} = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right],$$

    so $f'_{\hat{u}} = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right] \cdot [1, 3] = \boxed{\dfrac{4}{\sqrt{2}} \text{ or } 2\sqrt{2}}$.

89. Give a unit vector $\hat{u}$ such that $f'_{\hat{u}}(1, 1) = 0$ for $f(x, y) = x^3 y^4$.

    $\nabla f = \begin{bmatrix} 3x^2y^4 \\ 4x^3y^3 \end{bmatrix}$ and $\nabla f(1, 1) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. The unit vector in the same direction as this is $\frac{3}{5}\hat{\imath} + \frac{4}{5}\hat{\jmath}$, and the only unit vectors perpendicular to this are $\boxed{\begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix} \text{ or } \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}}$.

90. For $f(x, y) = 7y \sin(xy)$ at the point $(x, y) = (0, 2)$,

    (a) what are the smallest (most negative) possible value of $f'_{\hat{u}}(0, 2)$?

    $\nabla f(x, y) = \big[7y^2 \cos(xy), 7xy \cos(xy) + 7 \sin(xy)\big]$, so $\nabla f(0, 2) = [28, 0]$. The smallest possible value of $f'_{\hat{u}}(0, 2)$ is $\boxed{-28}$.

    (b) give the direction, as a unit vector, in which f decreases as much as possible, that is, the direction in which $f'_{\hat{u}}(0, 2)$ is most negative.

    The opposite direction from $\nabla f(0, 2) = [28, 0]$, which the same direction as $[-28, 0]$. As a unit vector, this is $\boxed{[-1, 0]}$.

    (c) give a direction, as a unit vector, in which the derivative of f is zero.

    Any direction perpendicular to $\nabla f(0, 2) = [28, 0]$, which will be $\boxed{[0, 1] \text{ or } [0, -1]}$.

91. Give the direction in which $\dfrac{x+y^2}{2e^x}$ increases the most from the point $(1,2)$.

This is the direction of $\nabla f(1,2) = \left[\frac{-2}{e}, \frac{2}{e}\right]$, which is North-West ↖. As a unit vector, this direction is $\boxed{\hat{u} = \left[\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]}$. The tasks does not specify that the direction should be given as a unit vector, so $\boxed{\text{any positive scalar multiple of } [-1,1]}$ is correct.

A **critical point** (or **CP**) of a function of multiple variables is a point in the domain of the function where all partial derivatives are zero or where at least one partial derivative is undefined.

92. If $(3,8)$ is a critical point of $f(x,y)$, what is the value of $f'_{\hat{u}}(3,8)$ in the direction $\hat{u} = [-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}]$? In the direction of $\vec{v} = [8,7]$? $\boxed{0}$ in any direction!

93. Find the critical point(s) of

$$f(x,y) = 2x^3 - 3x^2y - 12x^2 - 3y^2.$$

$\nabla f = \begin{bmatrix} 6x(-4+x-y) \\ -3(x^2+2y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ at three points: $\boxed{(0,0),\ (-4,-8),\ (2,-2)}$

94. Find the critical point(s) of each of the following functions.

(a) $f(x,y) = e^x - xy$. $\nabla f = \begin{bmatrix} e^x - y \\ -x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ when $(x,y) = \boxed{(0,1)}$ only.

(b) $f(x,y) = y\ln(x^2)$ $\boxed{(1,0) \text{ and } (-1,0)}$

(c) $f(x,y) = x\sin(y) + 9$ $\boxed{\text{All points } (0, n\pi) \text{ for } n \in \mathbb{Z}}$

(d) $f(x,y) = x^3 + 8y^3 - 3xy$ $\boxed{(0,0) \text{ and } (\frac{1}{2}, \frac{1}{4})}$

95. Find the critical point(s) of $f(x,y,z) = x\ln(z) + y^3z$. $\boxed{(0,0,1)}$

96. Find the critical point(s) of $f(x,y,z) = \frac{1}{3}x^3 - x + yz - y - z^2$. $\boxed{(-1,2,1) \text{ and } (1,2,1)}$

The **Hessian** of $f(x,y)$ is the matrix $\begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{bmatrix}$. We write $\mathbf{H}f$ for this matrix.

The Second Derivative Test: For each critical point of a function $f(x,y)$, calculate the "discriminant"

$$D = \det(\mathbf{H}f) = f''_{xx}\, f''_{yy} - (f''_{xy})^2.$$

If $D < 0$, then the CP is a **saddle** (also called a **saddle point**).
If $D > 0$ and $f''_{xx} > 0$, then the CP is a **local minimum**.
If $D > 0$ and $f''_{xx} < 0$, then the CP is a **local maximum**.
If $D = 0$, or if $D > 0$ but $f''_{xx} = 0$, then the test does not help classify the CP.

97. Calculate the Hessian of $f(x,y) = x\ln(xy)$ at the point $(3, \frac{1}{2})$.

Using **Task 61**, $\mathbf{H}f = \begin{bmatrix} 1/x & 1/y \\ 1/y & -x/y^2 \end{bmatrix}$, so $\mathbf{H}f(3, \frac{1}{2}) = \boxed{\begin{bmatrix} 1/3 & 2 \\ 2 & -12 \end{bmatrix}}$.

98. Calculate the determinant of the Hessian of $f = x^2 \sin(y)$ at the point $(4, \frac{\pi}{3})$.

$\mathbf{H}f(4, \frac{\pi}{3}) = \begin{bmatrix} \sqrt{3} & 4 \\ 4 & -8\sqrt{3} \end{bmatrix}$, so $\det\left(\mathbf{H}f(4, \frac{\pi}{3})\right) = \boxed{-40}$.

99. Find and classify all the critical point(s) of $f(x, y) = 2x^2 + y^2 - 3xy$.

$\nabla f = \begin{bmatrix} 4x - 3y \\ -3x + 2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ only at $(0, 0)$.

$\mathbf{H}f(0, 0) = \begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix}$ has determinant $8 - 9 = -1 < 0$, so $\boxed{(0,0) \text{ is a saddle.}}$

100. Find and classify all the critical point(s) of $f(x, y) = 2x^3 - 3x^2y - 12x^2 - 3y^2$.

From **Task 93** the CPs are $(0, 0)$, $(-4, -8)$, $(2, -2)$.

$H = \begin{bmatrix} 12x - 6y - 24 & -6x \\ -6x & -6 \end{bmatrix}$

$D = (12x - 6y - 24)(-6) - (-6x)^2 = 36(y - 2x - x^2 + 4)$.

x	y	D	f''_{xx}	
0	0	+	−	local max at $(0,0)$
−4	−8	−		saddle at $(-4,-8)$
2	−2	−		saddle at $(2,-2)$

101. Suppose $f(x, y)$ is a twice-differentiable function and that

$$\begin{aligned} &f(-3,0)=5, && f(4,9)=37, && f(1,-8)=-5,\\ &f'_x(-3,0)=0, && f'_x(4,9)=0, && f'_x(1,-8)=0,\\ &f'_y(-3,0)=1, && f'_y(4,9)=0, && f'_y(1,-8)=0,\\ &f''_{xx}(-3,0)=0, && f''_{xx}(4,9)=4, && f''_{xx}(1,-8)=1,\\ &f''_{xy}(-3,0)=-4, && f''_{xy}(4,9)=2, && f''_{xy}(1,-8)=2,\\ &f''_{yy}(-3,0)=12, && f''_{yy}(4,9)=11, && f''_{yy}(1,-8)=1. \end{aligned}$$

(a) Is $(-3, 0)$ a critical point of f? $\boxed{\text{No}}$ because $f'_y(-3, 0) \neq 0$. Is $(4, 9)$? $\boxed{\text{Yes.}}$ Is $(1, -8)$? $\boxed{\text{Yes.}}$

(b) Is $(-3, 0)$ a local minimum of f? $\boxed{\text{No}}$ because $(-3, 0)$ is not a critical point. Is $(4, 9)$? $\boxed{\text{Yes}}$ because $\nabla f(4, 9) = [0, 0]$ and $\det \mathbf{H}f(4, 9) = \det\left[\begin{smallmatrix} 4 & 2 \\ 2 & 11 \end{smallmatrix}\right] = 44 - 4 = 40 > 0$ and $f''_{xx}(4, 9) = 4 > 0$. Is $(1, -8)$? $\boxed{\text{No}}$ because $\det \mathbf{H}f(1, -8) = \det\left[\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right] = -3 < 0$.

(c) Is $(-3, 0)$ a local maximum of f? $\boxed{\text{No.}}$ Is $(4, 9)$? $\boxed{\text{No.}}$ Is $(1, -8)$? $\boxed{\text{No.}}$

(d) Is $(-3, 0)$ a saddle point of f? $\boxed{\text{No.}}$ Is $(4, 9)$? $\boxed{\text{No}}$ because $\det \mathbf{H}f(4, 9) > 0$. Is $(1, -8)$? $\boxed{\text{Yes}}$ because $\nabla f(1, -8) = [0, 0]$ and $\det \mathbf{H}f(1, -8) < 0$.

102. Find and classify the CP of the function $f(x, y)$ for which $\nabla f(x, y) = \begin{bmatrix} 3x^2 - 3y \\ 24y^2 - 3x \end{bmatrix}$.

The solutions to $\nabla f = \vec{0}$ are $(0,0)$ and $(\frac{1}{2},\frac{1}{4})$ (this is exactly **Task 94(d)**). Using $f'_x = 3x^2 - 3y$ and $f'_y = 24y^2 - 3x$, we get that

$$\begin{aligned} f''_{xx} &= \tfrac{\partial}{\partial x}\left[3x^2 - 3y\right] = 6x \\ f''_{yy} &= \tfrac{\partial}{\partial y}\left[24y^2 - 3x\right] = 48 \\ f''_{xy} &= \tfrac{\partial}{\partial y}\left[3x^2 - 3y\right] = -3 \text{ or } \tfrac{\partial}{\partial x}\left[24y^2 - 3x\right] = -3 \end{aligned}$$

And thus

$$\mathbf{H}f = \begin{bmatrix} 6x & -3 \\ -3 & 48 \end{bmatrix}, \qquad \det(\mathbf{H}f) = 288x - 9.$$

At the point $(0,0)$, we have $\det(\mathbf{H}f) = 0 - 9 < 0$, so $\boxed{(0,0) \text{ is a saddle}}$. At the point $(\frac{1}{2},\frac{1}{4})$, we have $\det(\mathbf{H}f) = 144 - 9 > 0$ and $f''_{xx} = 3 > 0$, so $\boxed{(\frac{1}{2},\frac{1}{4}) \text{ is a local min}}$.

103. Match each gradient vector with the Hessian matrix for the same function.

(a) $\nabla f = \begin{bmatrix} 3x^2 + y \\ x + 30y^2 \end{bmatrix}$ (I) $\mathbf{H}f = \begin{bmatrix} 90x^8 + 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y \end{bmatrix}$

(b) $\nabla f = \begin{bmatrix} 3x^2 + y \\ x + 15y^4 \end{bmatrix}$ (II) $\mathbf{H}f = \begin{bmatrix} 6x & 1 \\ 1 & 60y^3 \end{bmatrix}$

(c) $\nabla f = \begin{bmatrix} 10x^9 + 2xy^3 \\ 3x^2y^2 + 1 \end{bmatrix}$ (III) $\mathbf{H}f = \begin{bmatrix} 40x^3 + 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y \end{bmatrix}$

(d) $\nabla f = \begin{bmatrix} 10x^4 + 2xy^3 \\ 3x^2y^2 + 1 \end{bmatrix}$ (IV) $\mathbf{H}f = \begin{bmatrix} 6x & 1 \\ 1 & 60y \end{bmatrix}$

$\boxed{\text{a-IV, b-II, c-I, d-III}}$

104. Evaluate the iterated integral $\displaystyle\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) \mathrm{d}y\,\mathrm{d}x$ by following these steps:

(a) Calculate $\displaystyle\int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) \mathrm{d}y$. Your answer should be a formula involving x.

$$\left[\frac{y^2}{2x} + x\ln(y)\right]_{y=1}^{y=2} = \boxed{\frac{3}{2x} + x\ln(2)}$$

(b) Calculate $\displaystyle\int_1^4 f(x)\,\mathrm{d}x$, where $f(x)$ is the answer to part (a). The answer to (b) should be a number, and this is exactly the same as $\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) \mathrm{d}y\,\mathrm{d}x$.

$$\left[\tfrac{3}{2}\ln(x) + \tfrac{1}{2}\ln(2)x^2\right]_{x=1}^{x=4} = \boxed{\tfrac{21}{2}\ln(2)}$$

105. Evaluate (that is, find the value of) the following iterated integrals:

(a) $\int_0^1 \int_1^2 (4x^3 - 9x^2y^2)\, \mathrm{d}y\, \mathrm{d}x$.

$$\int_1^2 (4x^3 - 9x^2y^2)\, \mathrm{d}y = \Big[4x^3y - 3x^2y^3\Big]_{y=1}^{y=2}$$
$$= \left(8x^3 - 24x^2\right) - \left(4x^3 - 3x^2\right)$$
$$= 4x^3 - 21x^2$$
$$\int_0^1 \int_1^2 (4x^3 - 9x^2y^2)\, \mathrm{d}y\, \mathrm{d}x = \int_0^1 (4x^3 - 21x^2)\, \mathrm{d}x$$
$$= \Big[x^4 - 7x^3\Big]_{x=0}^{x=1}$$
$$= 1 - 7 = \boxed{-6}$$

(b) $\int_0^1 \int_1^2 (4x^3 - 9x^2y^2)\, \mathrm{d}x\, \mathrm{d}y$. $\int_0^1 \left(15 - 21y^2\right) \mathrm{d}y = \boxed{8}$

(c) $\int_1^2 \int_0^1 (4x^3 - 9x^2y^2)\, \mathrm{d}x\, \mathrm{d}y$. $\int_1^2 \left(1 - y^3\right) \mathrm{d}y = \boxed{-6}$

106. Calculate $\int_0^\pi \int_0^1 (\sin\theta)\, e^{r^2}\, r\, \mathrm{d}r\, \mathrm{d}\theta$. $\boxed{e - 1}$

107. Evaluate the iterated integral $\int_2^4 \int_1^y xy\, \mathrm{d}x\, \mathrm{d}y$ by following these steps:

(a) Calculate $\int_1^y xy\, \mathrm{d}x$. Your answer should be a formula involving y. $\boxed{\dfrac{y^3}{2} - \dfrac{y}{2}}$

(b) Calculate $\int_2^4 g(y)\, \mathrm{d}y$, where $g(y)$ is the answer to part (a). The answer to (b) should be a number, and this is exactly the same as $\int_2^4 \int_1^y xy\, \mathrm{d}x\, \mathrm{d}y$. $\boxed{27}$

108. Evaluate $\int_0^3 \int_0^{2y} \sin(\pi y^2)\, \mathrm{d}x\, \mathrm{d}y$. $\boxed{\dfrac{2}{\pi}}$

109. Calculate $\int_0^{\pi/2} \int_{x/2}^{\sqrt{\sin x}} 8y\, \mathrm{d}y\, \mathrm{d}x$.

$$\int_0^{\pi/2} \left(4y^2 \Big|_{y=x/2}^{y=\sqrt{\sin x}} \right) \mathrm{d}x = \int_0^{\pi/2} \left(4\sin(x) - x^2\right) \mathrm{d}x = \Big[-4\cos(x) + \tfrac{1}{3}x^3\Big]_{x=0}^{x=\pi/2} = \boxed{\frac{\pi^3}{24} - 4}$$