Analysis 2, Summer 2024

List 3

Directional derivatives, critical points

86. (a) Give the derivative of $\dfrac{x^3}{\sin(\pi y)}$ at the point $(4, \frac{1}{4})$ in the direction $\hat{\imath} = [1, 0]$.

    (b) Give the derivative of $\dfrac{x^3}{\sin(\pi y)}$ at the point $(4, \frac{1}{4})$ in the direction $\hat{\jmath} = [0, 1]$.

> The **directional derivative** of $f(x, y)$ at the point (a, b) in the direction of the **unit** vector $\hat{u}$ (a vector of length 1) is written as $f'_{\hat{u}}(a, b)$ and can be calculated as
> $$f'_{\hat{u}}(a, b) = \nabla f(a, b) \cdot \hat{u}.$$

87. For $f(x, y) = x^2 \sin(y)$, calculate the directional derivative at $(4, \frac{\pi}{3})$ in the direction $\hat{u} = \left[\frac{\sqrt{3}}{2}, \frac{1}{2}\right]$.

88. What is the derivative of $f(x, y) = xe^y$ at the point $(3, 0)$ in the direction $[1, 1]$?

89. Give a unit vector $\hat{u}$ such that $f'_{\hat{u}}(1, 1) = 0$ for $f(x, y) = x^3y^4$.

90. For $f(x, y) = 7y \sin(xy)$ at the point $(x, y) = (0, 2)$,
    (a) what are the smallest (most negative) possible value of $f'_{\hat{u}}(0, 2)$?
    (b) give the direction, as a unit vector, in which f decreases as much as possible, that is, the direction in which $f'_{\hat{u}}(0, 2)$ is most negative.
    (c) give a direction, as a unit vector, in which the derivative of f is zero.

91. Give the direction in which $\dfrac{x + y^2}{2e^x}$ increases the most from the point $(1, 2)$.

> A **critical point** (or **CP**) of a function of multiple variables is a point in the domain of the function where all partial derivatives are zero or where at least one partial derivative is undefined.

92. If $(3, 8)$ is a critical point of $f(x, y)$, what is the value of $f'_{\hat{u}}(3, 8)$ in the direction $\hat{u} = [-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}]$? In the direction of $\vec{v} = [8, 7]$?

93. Find the critical point(s) of
$$f(x, y) = 2x^3 - 3x^2y - 12x^2 - 3y^2.$$

94. Find the critical point(s) of each of the following functions.
    (a) $f(x, y) = e^x - xy$
    (b) $f(x, y) = y \ln(x^2)$
    (c) $f(x, y) = x \sin(y) + 9$
    (d) $f(x, y) = x^3 + 8y^3 - 3xy$

95. Find the critical point(s) of $f(x, y, z) = x \ln(z) + y^3 z$.

96. Find the critical point(s) of $f(x, y, z) = \frac{1}{3}x^3 - x + yz - y - z^2$.

The **Hessian** of $f(x,y)$ is the matrix $\begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{bmatrix}$. We write $\mathbf{H}f$ for this matrix.

The Second Derivative Test: For each critical point of a function $f(x,y)$, calculate the "discriminant"

$$D = \det(\mathbf{H}f) = f''_{xx}\, f''_{yy} - (f''_{xy})^2.$$

If $D < 0$, then the CP is a **saddle** (also called a **saddle point**).
If $D > 0$ and $f''_{xx} > 0$, then the CP is a **local minimum**.
If $D > 0$ and $f''_{xx} < 0$, then the CP is a **local maximum**.
If $D = 0$, or if $D > 0$ but $f''_{xx} = 0$, then the test does not help classify the CP.

97. Calculate the Hessian of $f(x,y) = x\ln(xy)$ at the point $(3, \frac{1}{2})$.

98. Calculate the determinant of the Hessian of $f = x^2\sin(y)$ at the point $(4, \frac{\pi}{3})$.

99. Find and classify all the critical point(s) of $f(x,y) = 2x^2 + y^2 - 3xy$.

100. Find and classify all the critical point(s) of $f(x,y) = 2x^3 - 3x^2y - 12x^2 - 3y^2$.

101. Suppose $f(x,y)$ is a twice-differentiable function and that

$$\begin{array}{ccc} f(-3,0) = 5, & f(4,9) = 37, & f(1,-8) = -5, \\ f'_x(-3,0) = 0, & f'_x(4,9) = 0, & f'_x(1,-8) = 0, \\ f'_y(-3,0) = 1, & f'_y(4,9) = 0, & f'_y(1,-8) = 0, \\ f''_{xx}(-3,0) = 0, & f''_{xx}(4,9) = 4, & f''_{xx}(1,-8) = 1, \\ f''_{xy}(-3,0) = -4, & f''_{xy}(4,9) = 2, & f''_{xy}(1,-8) = 2, \\ f''_{yy}(-3,0) = 12, & f''_{yy}(4,9) = 11, & f''_{yy}(1,-8) = 1. \end{array}$$

(a) Is $(-3,0)$ a critical point of f? Is $(4,9)$? Is $(1,-8)$?
(b) Is $(-3,0)$ a local minimum of f? Is $(4,9)$? Is $(1,-8)$?
(c) Is $(-3,0)$ a local maximum of f? Is $(4,9)$? Is $(1,-8)$?
(d) Is $(-3,0)$ a saddle point of f? Is $(4,9)$? Is $(1,-8)$?

102. Find and classify the CP of the function $f(x,y)$ for which $\nabla f(x,y) = \begin{bmatrix} 3x^2 - 3y \\ 24y^2 - 3x \end{bmatrix}$.

103. Match each gradient vector with the Hessian matrix for the same function.

(a) $\nabla f = \begin{bmatrix} 3x^2 + y \\ x + 30y^2 \end{bmatrix}$ (I) $\mathbf{H}f = \begin{bmatrix} 90x^8 + 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y \end{bmatrix}$

(b) $\nabla f = \begin{bmatrix} 3x^2 + y \\ x + 15y^4 \end{bmatrix}$ (II) $\mathbf{H}f = \begin{bmatrix} 6x & 1 \\ 1 & 60y^3 \end{bmatrix}$

(c) $\nabla f = \begin{bmatrix} 10x^9 + 2xy^3 \\ 3x^2y^2 + 1 \end{bmatrix}$ (III) $\mathbf{H}f = \begin{bmatrix} 40x^3 + 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y \end{bmatrix}$

(d) $\nabla f = \begin{bmatrix} 10x^4 + 2xy^3 \\ 3x^2y^2 + 1 \end{bmatrix}$ (IV) $\mathbf{H}f = \begin{bmatrix} 6x & 1 \\ 1 & 60y \end{bmatrix}$

104. Evaluate the iterated integral $\displaystyle\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) \mathrm{d}y\, \mathrm{d}x$ by following these steps:

(a) Calculate $\displaystyle\int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) \mathrm{d}y$. Your answer should be a formula involving x.

(b) Calculate $\displaystyle\int_1^4 f(x)\, \mathrm{d}x$, where $f(x)$ is the answer to part (a). The answer to (b) should be a number, and this is exactly the same as $\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) \mathrm{d}y\, \mathrm{d}x$.

105. Evaluate (that is, find the value of) the following iterated integrals:

(a) $\displaystyle\int_0^1 \int_1^2 (4x^3 - 9x^2y^2)\, \mathrm{d}y\, \mathrm{d}x$.

(b) $\displaystyle\int_0^1 \int_1^2 (4x^3 - 9x^2y^2)\, \mathrm{d}x\, \mathrm{d}y$.

(c) $\displaystyle\int_1^2 \int_0^1 (4x^3 - 9x^2y^2)\, \mathrm{d}x\, \mathrm{d}y$.

106. Calculate $\displaystyle\int_0^\pi \int_0^1 (\sin\theta)\, e^{r^2}\, r\, \mathrm{d}r\, \mathrm{d}\theta$.

107. Evaluate the iterated integral $\displaystyle\int_2^4 \int_1^y xy\, \mathrm{d}x\, \mathrm{d}y$ by following these steps:

(a) Calculate $\displaystyle\int_1^y xy\, \mathrm{d}x$. Your answer should be a formula involving y.

(b) Calculate $\displaystyle\int_2^4 g(y)\, \mathrm{d}y$, where $g(y)$ is the answer to part (a). The answer to (b) should be a number, and this is exactly the same as $\int_2^4 \int_1^y xy\, \mathrm{d}x\, \mathrm{d}y$.

108. Evaluate $\displaystyle\int_0^3 \int_0^{2y} \sin(\pi y^2)\, \mathrm{d}x\, \mathrm{d}y$.

109. Calculate $\displaystyle\int_0^{\pi/2} \int_{x/2}^{\sqrt{\sin x}} 8y\, \mathrm{d}y\, \mathrm{d}x$.