Analysis 2, Summer 2024

List 4

Double integrals

110. Calculate either one of the integrals

$$\int_1^2\int_0^4 (y^3+xy)\,\mathrm{d}x\,\mathrm{d}y \qquad \text{or} \qquad \int_0^4\int_1^2 (y^3+xy)\,\mathrm{d}y\,\mathrm{d}x.$$

(The answers are the same since these both describe the same integral over a rectangle.)

$$\int_1^2\int_0^4 (y^3+xy)\,\mathrm{d}x\,\mathrm{d}y = \int_1^2 \Big(4y^3+8y\Big)\,\mathrm{d}y = \boxed{27} \text{ or}$$
$$\int_0^4\int_1^2 (y^3+xy)\,\mathrm{d}y\,\mathrm{d}x = \int_0^4 \left(\frac{3x}{2}+\frac{15}{4}\right)\mathrm{d}x = \boxed{27}$$

111. Calculate the integral over a rectangle (you may choose whether to integrate $\mathrm{d}y\,\mathrm{d}x$ or $\mathrm{d}x\,\mathrm{d}y$):

(a) $\displaystyle\iint_R \left(\frac{x}{y}+\frac{y}{x}\right)\mathrm{d}A$ with $R=\{(x,y): 1\le x\le 4 \text{ and } 1\le y\le 2\}$.

$$\int_1^2 \left(\frac{x}{y}+\frac{y}{x}\right)\mathrm{d}y = \left[\frac{y^2}{2x}+x\ln y\right]_{y=1}^{y=2}$$
$$= \left(\frac{4}{2x}+x\ln 2\right)-\left(\frac{1}{2x}+x\ln 1\right)$$
$$= \frac{3}{2x}+x\ln 2$$
$$\int_1^4\int_1^2 \left(\frac{x}{y}+\frac{y}{x}\right)\mathrm{d}y\,\mathrm{d}x = \int_1^4\left(\frac{3}{2x}+x\ln 2\right)\mathrm{d}x = \left[\frac{x^2}{2}\ln 2+\frac{3}{2}\ln x\right]_{x=1}^{x=4}$$
$$= \left(8\ln 2+\frac{3}{2}\ln 4\right)-\left(\frac{1}{2}\ln 2+\frac{3}{2}\ln 1\right) = \boxed{\frac{21}{2}\ln 2}$$

$$\text{or } \int_1^2\left(\int_1^4\left(\frac{x}{y}+\frac{y}{x}\right)\mathrm{d}x\right)\mathrm{d}y = \int_1^4\left(\frac{15}{2y}+y\ln 4\right)\mathrm{d}y = \boxed{\frac{21}{2}\ln 2}$$

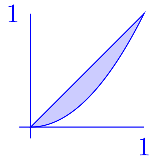
(b) $\displaystyle\iint_R x\sin(xy)\,\mathrm{d}A$ with $R=[0,1]\times[\pi,2\pi]$.

$$\int_0^1\int_\pi^{2\pi} x\sin(xy)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \Big[-\cos(xy)\Big]_{y=\pi}^{y=2\pi}\mathrm{d}y\,\mathrm{d}x$$
$$= \int_0^1 \Big(-\cos(2\pi x)+\cos(\pi x)\Big)\mathrm{d}y\,\mathrm{d}x = \boxed{0}$$

(c) $\displaystyle\iint_R \frac{x+y}{e^x}\,\mathrm{d}A$, where R has $(0,0)$ at the bottom-left and $(1,1)$ at top-right.

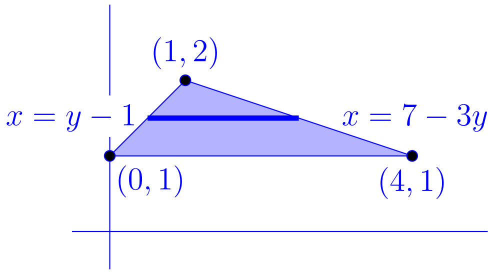
$$\int_0^1\int_0^1 \frac{x+y}{e^x}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\left[\frac{x+y+1}{-e^{-x}}\right]_{x=0}^{x=1}\mathrm{d}y = \int_0^1\left(1-\tfrac{2}{e}+y-\tfrac{y}{e}\right)\mathrm{d}y = \boxed{\frac{3}{2}-\frac{5}{2e}}$$

112. Draw the domain of integration for $\displaystyle\int_0^1 \int_{x^2}^{x} \frac{y}{x}\, dy\, dx$ and evaluate the integral.



$$\int_0^1 \left[\frac{y^2}{2x}\right]_{y=x^2}^{y=x} dx = \int_0^1 \left(\frac{x}{2} - \frac{x^3}{2}\right) dx = \left[\frac{x^2}{4} - \frac{x^4}{8}\right]_{x=0}^{x=1} = \boxed{\frac{1}{8}}$$

113. Integrate $f(x, y) = y^2$ over the triangle with vertices at $(0, 1)$, $(1, 2)$, and $(4, 1)$.



$$\int_1^2 \int_{y-1}^{7-3y} y^2\, dx\, dy$$

$$\int_{y-1}^{7-3y} y^2\, dx = \left[xy^2\right]_{x=y-1}^{x=7-3y} = (7-3y)y^2 - (y-1)y^2 = 8y^2 - 4y^3$$

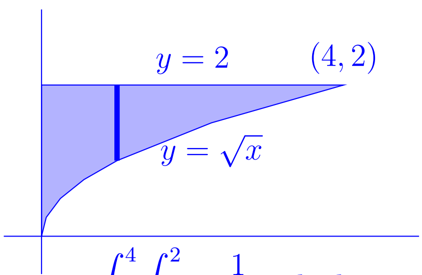
$$\int_1^2 \int_{y-1}^{7-3y} y^2\, dx\, dy = \int_1^2 (8y^2 - 4y^3)\, dy = \left[\frac{8}{3}y^3 - y^4\right]_{y=1}^{y=2}$$

$$= \frac{64}{3} - 16 - \frac{8}{3} + 1 = \boxed{\frac{11}{3}}$$

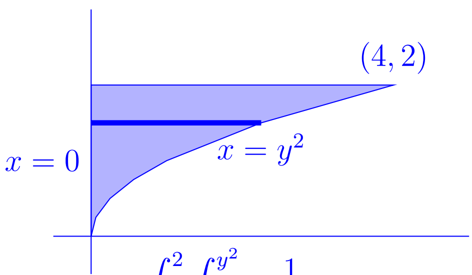
114. The integral $\displaystyle\int \frac{1}{y^3+1}\, dy$ is very difficult, so evaluate

$$\int_0^4 \int_{\sqrt{x}}^{2} \frac{1}{y^3+1}\, dy\, dx$$

by first "reversing the order of integration", that is, by changing to an integral $dx\, dy$ over the same region.



$$\int_0^4 \int_{\sqrt{x}}^{2} \frac{1}{y^3+1}\, dy\, dx$$



$$\int_0^2 \int_0^{y^2} \frac{1}{y^3+1}\, dx\, dy$$

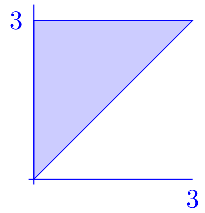
$$\int_0^{y^2} \frac{1}{y^3+1}\,\mathrm{d}x = \left[\frac{x}{y^3+1}\right]_{x=0}^{x=y^2} = \frac{y^2}{y^3+1}$$

$$\int_0^2 \int_0^{y^2} \frac{1}{y^3+1}\,\mathrm{d}x\,\mathrm{d}y = \int_0^2 \frac{y^2}{y^3+1}\,\mathrm{d}y = \int_1^9 \frac{1}{3u}\,\mathrm{d}u, \qquad \begin{aligned} u &= y^3+1 \\ \mathrm{d}u &= 3y^2\,\mathrm{d}y \end{aligned}$$

$$= \left[\frac{1}{3}\ln u\right]_{u=1}^{u=9} = \boxed{\frac{1}{3}\ln 9 = \frac{2}{3}\ln 3}$$

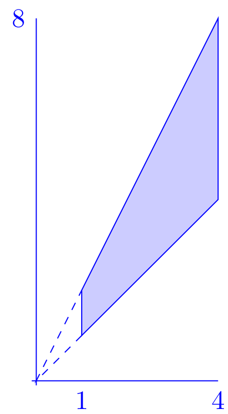
115. Draw the domain of integration and evaluate the integral:

(a) $\int_0^3 \int_0^y xy\,\mathrm{d}x\,\mathrm{d}y = \int_0^3 \left[\frac{x^2 y}{2}\right]_{x=0}^{x=y} \mathrm{d}y = \int_0^3 \frac{y^3}{2}\,\mathrm{d}y = \left[\frac{y^4}{8}\right]_{y=0}^{y=3} = \boxed{\frac{81}{8}}$

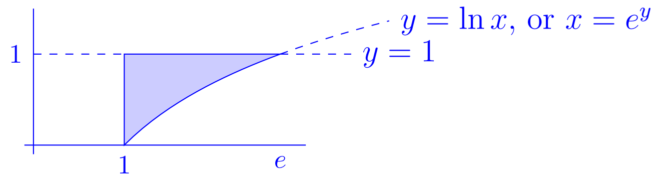


(b) $\int_1^4 \int_x^{2x} x^2\sqrt{y-x}\,\mathrm{d}y\,\mathrm{d}x$

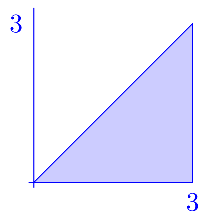
$= \int_1^4 \left[\tfrac{3}{2}x^2(y-x)^{3/2}\right]_{y=x}^{y=2x} \mathrm{d}x = \int_1^4 \tfrac{2}{3}x^{7/2}\,\mathrm{d}x = \boxed{\frac{2044}{27}}$



(c) $\int_1^e \int_{\ln x}^1 \frac{1}{e^y - 1}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_1^{e^y} \frac{1}{e^y-1}\,\mathrm{d}x\,\mathrm{d}y = \boxed{1}$



(d) $\int_0^3 \int_y^3 \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x\,\mathrm{d}y = \boxed{\sqrt{10}-1}$ Note: $\int_0^3 \int_0^x \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}y\,\mathrm{d}x$ is easier, but both orders are possible.



116. *Set up* each of the following as an iterated integral $\mathrm{d}x\,\mathrm{d}y$ or as an iterated integral $\mathrm{d}y\,\mathrm{d}x$.

(a) $\iint_R f\,\mathrm{d}A$, where R is the rectangle with corners $(3,0)$ and $(10,5)$.

$\boxed{\int_0^5 \int_3^{10} f\,\mathrm{d}x\,\mathrm{d}y}$ or $\boxed{\int_3^{10} \int_0^5 f\,\mathrm{d}y\,\mathrm{d}x}$

(b) $\iint_T f\,\mathrm{d}A$, where T is the triangle with corners $(0,0)$, $(-4,4)$, and $(4,4)$.

$\boxed{\int_0^4 \int_{-y}^{y} f\,\mathrm{d}x\,\mathrm{d}y}$ or $\boxed{\left(\int_{-4}^0 \int_{-x}^4 f\,\mathrm{d}y\,\mathrm{d}x\right) + \left(\int_0^4 \int_x^4 f\,\mathrm{d}y\,\mathrm{d}x\right)}$

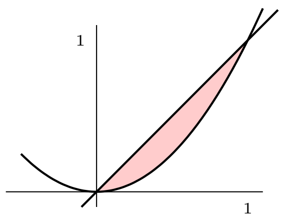
(c) $\iint_D f\,\mathrm{d}A$, where D is bounded by $y = x$ and $y = 2 - x^2$. $\boxed{\int_{-2}^1 \int_x^{2-x^2} f\,\mathrm{d}y\,\mathrm{d}x}$

(d) $\iint_D f\,\mathrm{d}A$, where D is bounded by $y = -2$, $y = \frac{1}{x}$, $y = -\sqrt{-x}$. $\boxed{\int_{-2}^{-1} \int_{-y^2}^{\frac{1}{y}} f\,\mathrm{d}x\,\mathrm{d}y}$

(e) $\iint_R f\,\mathrm{d}A$, where R is bounded by $y = x+3$ and $y = x^2+3x+3$. $\boxed{\int_{-2}^0 \int_{x^2+3x+3}^{x+3} f\,\mathrm{d}y\,\mathrm{d}x}$

(f) $\iint_R f\,\mathrm{d}A$, where R is bounded by $y = \sqrt{x}$, $x = 0$, $y = 1$. $\boxed{\int_0^1 \int_0^{y^2} f\,\mathrm{d}x\,\mathrm{d}y}$

☆117. Calculate the area of the region below by four different methods.



(a) as the area between curves $y = x^2$ and $y = x$: $\int_0^1 (x - x^2)\,\mathrm{d}x$ (Analysis I),

$\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{x=0}^{x=1} = \left(\frac{1}{2} - \frac{1}{3}\right) - 0 = \boxed{\frac{1}{6}}$.

(b) as the area between curves $x = \sqrt{y}$ and $x = y$: $\int_0^1 (\sqrt{y} - y)\,\mathrm{d}y$ (Analysis I),

$\left[\frac{2}{3}y^{3/2} - \frac{1}{2}y^2\right]_{y=0}^{y=1} = \left(\frac{2}{3} - \frac{1}{2}\right) - 0 = \boxed{\frac{1}{6}}$.

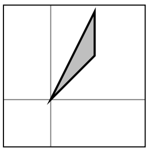
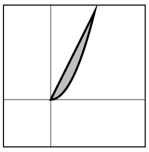
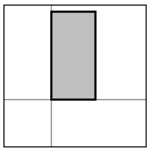
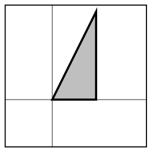
(c) as the integral of $f(x,y) = 1$ over this domain: $\int_0^1 \int_{x^2}^{x} 1\,\mathrm{d}y\,\mathrm{d}x$,

$\int_0^1 \left[y\right]_{y=x^2}^{y=x} \mathrm{d}x = \int_0^1 (x - x^2)\,\mathrm{d}x$. By part (a), this is $\boxed{\frac{1}{6}}$.

(d) as the integral of $f(x,y)=1$ over this domain: $\int_0^1 \int_y^{\sqrt{y}} 1\,\mathrm{d}x\,\mathrm{d}y$.

$\int_0^1 \left[x\right]_{x=y}^{x=\sqrt{y}} \mathrm{d}y = \int_0^1 (\sqrt{y}-y)\,\mathrm{d}y$. By part (b), this is $\boxed{\frac{1}{6}}$.

118. Which region below corresponds to $\int_0^2 \int_x^{2x} x\,\mathrm{d}y\mathrm{d}x$?

(A)  (B)  (C)  (D) 

119. Integrate

$$f(x,y)=e^{x/y}$$

over the region bounded by $y=\sqrt{x}$ and $x=0$ and $y=1$.

$\int_0^1 \int_0^{y^2} e^{x/y}\,\mathrm{d}x\,\mathrm{d}y = \boxed{\frac{1}{2}}$.

Note: although $\int_0^1 \int_{\sqrt{x}}^1 e^{x/y}\,\mathrm{d}y\,\mathrm{d}x$ is also $\frac{1}{2}$, the anti-derivative $\int e^{x/y}\,\mathrm{d}y$ does not have any elementary formula.

120. Re-write $\int_1^4 \int_x^{4x} x\sqrt{y-x}\,\mathrm{d}y\,\mathrm{d}x$ as the sum of two integrals $\mathrm{d}x\,\mathrm{d}y$.

$\boxed{\int_1^4 \int_1^y f\,\mathrm{d}x\,\mathrm{d}y + \int_4^{16} \int_{y/4}^4 f\,\mathrm{d}x\,\mathrm{d}y}$ (The value is $\frac{156\sqrt{3}}{35} + \frac{2384\sqrt{3}}{35} = \frac{508\sqrt{3}}{7}$.)

☆121. Calculate $\int_1^4 \int_x^{x+2} \int_0^{y^2} \frac{x+z}{y^2}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$.

Inner-most: $\int_0^{y^2} \frac{x+z}{y^2}\,\mathrm{d}z = \left.\frac{xz+\frac{1}{2}z^2}{y^2}\right|_{z=0}^{z=y^2} = x + \frac{y^2}{2}$

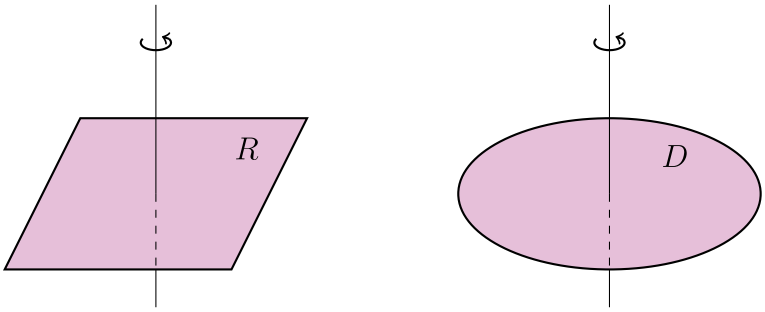
Middle: $\int_x^{x+2} \left(x+\frac{y^2}{2}\right)\mathrm{d}y = \left[xy+\frac{y^3}{6}\right]_{y=x}^{y=x+2} = x^2+4x+\frac{4}{3}$.

Outer: $\int_1^4 \left(x^2+4x+\frac{4}{3}\right)\mathrm{d}x = \left[\frac{x^3}{3}+2x^2+\frac{4x}{3}\right]_{x=1}^{x=4} = \frac{176}{3} - \frac{11}{3} = \boxed{55}$.

122. Does it take more energy to spin a disk or a square of the same mass around its center? The "moment of inertia"[1] of a shape is the integral of x^2+y^2 over that region. Calculate this number for both these regions:

$$\begin{array}{rl} \text{square} & R=\{(x,y): -1\le x\le 1, -1\le y\le 1\} \\ \text{disk} & D=\{(x,y): x^2+y^2\le \frac{4}{\pi}\}. \end{array}$$

The shape with higher moment of inertia will require more energy[2] to spin.



$$I_{\text{square}}=\int_{-1}^{1}\int_{-1}^{1}(x^2+y^2)\,\mathrm{d}y\,\mathrm{d}x=\int_{-1}^{1}(2x^2+\tfrac{2}{3})\,\mathrm{d}x=\boxed{\frac{8}{3}}.$$

$$I_{\text{disk}}=\int_{0}^{2\pi}\int_{0}^{2/\sqrt{\pi}}(r^2)r\,\mathrm{d}r\,\mathrm{d}\theta=\int_{0}^{2\pi}\frac{4}{\pi^2}\,\mathrm{d}\theta=\boxed{\frac{8}{\pi}}.$$

Since $3<\pi$, we have $\frac{1}{3}>\frac{1}{\pi}$ and so $I_{\text{square}}>I_{\text{disk}}$. Thus it takes more energy to spin a square.

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[1]Moment of inertia depends on density and is actually $I=\iint_S(x^2+y^2)\cdot\rho(x,y)\,\mathrm{d}A$ in general. Task 122 assumes the shapes have constant density $\rho(x,y)=1$. Also, this formula is for spinning around the origin. If an object is spun around a different point or around an axis instead of a point, the formula is slightly different.

[2]The energy required to get a still object to spin with angular velocity ω is $K=\frac{1}{2}I\omega^2$.