Analysis 2, Summer 2024

# List 4

*Double integrals*

110. Calculate either one of the integrals

$$\int_1^2\int_0^4 (y^3 + xy)\,\mathrm{d}x\,\mathrm{d}y \qquad \text{or} \qquad \int_0^4\int_1^2 (y^3 + xy)\,\mathrm{d}y\,\mathrm{d}x.$$

(The answers are the same since these both describe the same integral over a rectangle.)

111. Calculate the integral over a rectangle (you may choose whether to integrate $\mathrm{d}y\,\mathrm{d}x$ or $\mathrm{d}x\,\mathrm{d}y$):

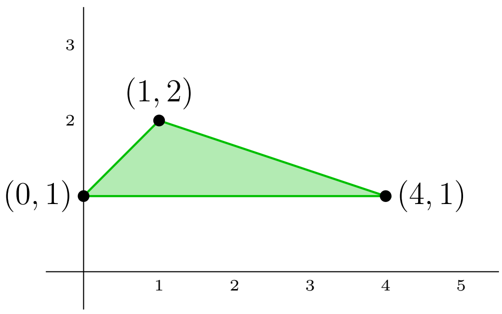
(a) $\displaystyle\iint_R \left(\frac{x}{y} + \frac{y}{x}\right) \mathrm{d}A$ with $R = \{(x, y) : 1 \leq x \leq 4 \text{ and } 1 \leq y \leq 2\}$.

(b) $\displaystyle\iint_R x\sin(xy)\,\mathrm{d}A$ with $R = [0, 1] \times [\pi, 2\pi]$.

(c) $\displaystyle\iint_R \frac{x+y}{e^x}\,\mathrm{d}A$, where $R$ has $(0, 0)$ at the bottom-left and $(1, 1)$ at top-right.

112. Draw the domain of integration for $\displaystyle\int_0^1\int_{x^2}^{x} \frac{y}{x^2}\,\mathrm{d}y\,\mathrm{d}x$ and evaluate the integral.

113. Integrate $f(x, y) = y^2$ over the triangle with vertices at $(0, 1)$, $(1, 2)$, and $(4, 1)$.




114. The integral $\displaystyle\int \frac{1}{y^3+1}\,\mathrm{d}y$ is very difficult, so evaluate

$$\int_0^4\int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,\mathrm{d}y\,\mathrm{d}x$$

by first “reversing the order of integration”, that is, by changing to an integral $\mathrm{d}x\,\mathrm{d}y$ over the same region.

115. Draw the domain of integration and evaluate the integral:

(a) $\displaystyle\int_0^3\int_0^y xy\,\mathrm{d}x\,\mathrm{d}y$

(b) $\displaystyle\int_1^4\int_x^{2x} x^2\sqrt{y-x}\,\mathrm{d}y\,\mathrm{d}x$

(c) $\displaystyle\int_1^e\int_{\ln x}^1 \frac{1}{e^y - 1}\,\mathrm{d}y\,\mathrm{d}x$

(d) $\displaystyle\int_0^3\int_y^3 \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x\,\mathrm{d}y$

116. *Set up* each of the following as an iterated integral $\mathrm{d}x\,\mathrm{d}y$ or as an iterated integral $\mathrm{d}y\,\mathrm{d}x$.

(a) $\iint_R f\,\mathrm{d}A$, where $R$ is the rectangle with corners $(3,0)$ and $(10,5)$.

(b) $\iint_T f\,\mathrm{d}A$, where $T$ is the triangle with corners $(0,0)$, $(-4,4)$, and $(4,4)$.

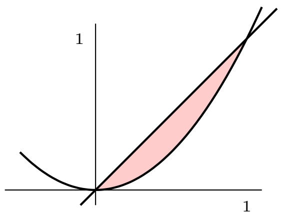
(c) $\iint_D f\,\mathrm{d}A$, where $D$ is bounded by $y = x$ and $y = 2 - x^2$.

(d) $\iint_D f\,\mathrm{d}A$, where $D$ is bounded by $y = -2, y = \frac{1}{x}, y = -\sqrt{-x}$.

(e) $\iint_R f\,\mathrm{d}A$, where $R$ is bounded by $y = x + 3$ and $y = x^2 + 3x + 3$.

(f) $\iint_R f\,\mathrm{d}A$, where $R$ is bounded by $y = \sqrt{x}, x = 0, y = 1$.

☆117. Calculate the area of the region below by four different methods.



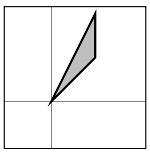
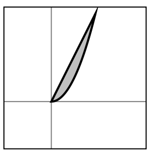
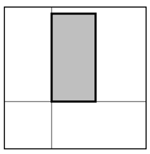
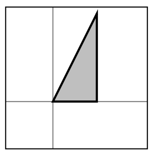

(a) as the area between curves $y = x^2$ and $y = x$: $\int_0^1 (x - x^2)\,\mathrm{d}x$ (Analysis I),

(b) as the area between curves $x = \sqrt{y}$ and $x = y$: $\int_0^1 (\sqrt{y} - y)\,\mathrm{d}y$ (Analysis I),

(c) as the integral of $f(x,y) = 1$ over this domain: $\int_0^1 \int_{x^2}^{x} 1\,\mathrm{d}y\,\mathrm{d}x$,

(d) as the integral of $f(x,y) = 1$ over this domain: $\int_0^1 \int_y^{\sqrt{y}} 1\,\mathrm{d}x\,\mathrm{d}y$.

118. Which region below corresponds to $\int_0^2 \int_x^{2x} x\,\mathrm{d}y\mathrm{d}x$?

(A)  (B)  (C)  (D) 

119. Integrate

$$f(x,y) = e^{x/y}$$

over the region bounded by $y = \sqrt{x}$ and $x = 0$ and $y = 1$.

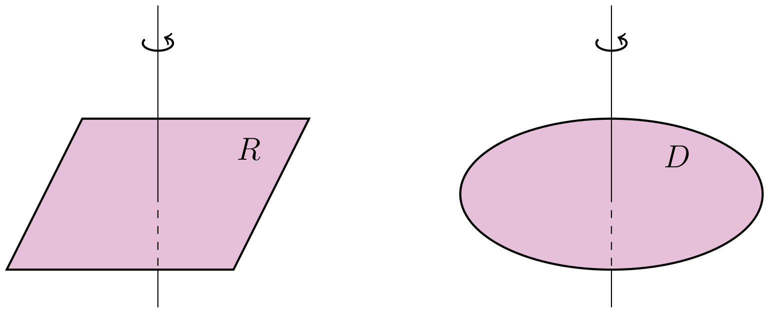
120. Re-write $\displaystyle\int_1^4 \int_x^{4x} x\sqrt{y-x}\,\mathrm{d}y\,\mathrm{d}x$ as the sum of two integrals $\mathrm{d}x\,\mathrm{d}y$.

☆121. Calculate $\displaystyle\int_1^4 \int_x^{x+2} \int_0^{y^2} \frac{x+z}{y^2}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$.

122. Does it take more energy to spin a disk or a square of the same mass around its center? The "moment of inertia"[1] of a shape is the integral of $x^2 + y^2$ over that region. Calculate this number for both these regions:

$$\begin{array}{cc} \text{square} & R = \{(x,y) : -1 \le x \le 1, -1 \le y \le 1\} \\ \text{disk} & D = \{(x,y) : x^2 + y^2 \le \frac{4}{\pi}\}. \end{array}$$

The shape with higher moment of inertia will require more energy[2] to spin.




[1]Moment of inertia depends on density and is actually $I = \iint_S (x^2+y^2)\cdot\rho(x,y)\,\mathrm{d}A$ in general. Task 122 assumes the shapes have constant density $\rho(x,y) = 1$. Also, this formula is for spinning around the origin. If an object is spun around a different point or around an axis instead of a point, the formula is slightly different.

[2]The energy required to get a still object to spin with angular velocity $\omega$ is $K = \frac{1}{2}I\omega^2$.