Analysis 2, Summer 2024

List 6

Vocabulary, direct ODEs/PDEs/IVPs

150. If $f'(x) = x$ and $f(2) = 3$, what is the function $f(x)$? $\boxed{f(x) = \frac{1}{2}x^2 + 1}$

An **ordinary differential equation** (ODE) is an equation that includes a derivative of a function with one variable. The **order** of an ODE is the highest derivative that appears in the equation.

151. Give the order of each of the following differential equations:
    (a) $xy' + x^2 = 0$ $\boxed{\text{order 1}}$
    (b) $y'' = t$ $\boxed{\text{order 2}}$
    (c) $\cos(y') = \sin(y)$ $\boxed{\text{order 1}}$
    (d) $(y')^3 = e^t$ $\boxed{\text{order 1}}$
    (e) $y \cdot y'' = \sin(t)$ $\boxed{\text{order 2}}$

152. Match each ODE to its solution:

(a) $y'(x) = x + y$ $\boxed{\text{(IV)}}$	(I) $y(x) = \dfrac{1}{\cos(x) - C}$
(b) $y'(x) = 2xy$ $\boxed{\text{(II)}}$	(II) $y(x) = Ce^{(x^2)}$
(c) $y'(x) = \sin(x) \cdot y^2$ $\boxed{\text{(I)}}$	(III) $y(x) = \dfrac{\pm 1}{\sqrt{2x - C}}$
(d) $y'(x) = -y^3$ $\boxed{\text{(III)}}$	(IV) $y(x) = Ce^x - x - 1$

153. Which of the following satisfy $x'' = x$?
    (a) $x = 5e^t$
    (b) $x = e^t + 9e^{-t}$
    (c) $\ln(x) = 12t$
    (d) $x = C_1e^t + C_2e^{-t}$
    (e) $2x = 19e^{-t}$
    (f) $xe^t = C_1e^{2t} + C_2$

    $\boxed{\text{All of them!}}$

A **partial differential equation** (PDE) is an equation that includes a partial derivative.

154. Solve the PDE $f'_x = 2x + y^2e^{xy^2}$, $f'_y = 2xye^{xy^2} + 9y^2$.

    See **Task 146**. Based on f'_x, the function f will be some

$$f = x^2 + e^{xy^2} + g(y),$$

where $g(y)$ is a function of y but is constant with respect to x (and so does not affect f'_x at all). Therefore

$$f'_y = 0 + e^{xy^2}(2yx) + g',$$

so in order to match f'_y from the task it must be that $g' = 9y^2$. Therefore $g = 3y^3 + C$, and

$$\boxed{f = x^2 + e^{xy^2} + 3y^3 + C}$$

describes all functions with the given gradient.

155. For each PDE below, state whether any solution $f(x, y)$ exists.

(a) $f'_x = 5, f'_y = 6$ $\boxed{\text{Yes}}$ because $f''_{xy} = 0 = f''_{yx}$.

(b) $f'_x = 5x, f'_y = 6y$ $\boxed{\text{Yes}}$

(c) $f'_x = 5y, f'_y = 6x$ $\boxed{\text{No}}$ because f''_{xy} would be 5 but f''_{yx} would be 6.

(d) $f'_x = 5y, f'_y = 5x$ $\boxed{\text{Yes}}$

156. Give solutions for the PDEs from Task 155 that have solutions.

(a) $\boxed{f = 5x + 6y + C}$

(b) $\boxed{f = \frac{5}{2}x^2 + 3y^2 + C}$

(d) $\boxed{f = 5xy + C}$

A **general solution** to an ODE describes all possible solutions to that differential equation, using one of more "constants of integration". A **particular solution** to an ODE is one specific function (*not* using C or C_1, C_2, etc.).

157. Give three different particular solutions to $y' = 11x^4$. See **Task 31a**. These must all be of the form $y = \frac{11}{5}x^5 + C$. Three examples are $y = \frac{11}{5}x^5$, $y = \frac{11}{5}x^5 + 382$, and $y = \frac{11}{5}x^5 - \pi$, but you could have other examples.

158. Give the general solution to $y' = \sin(x)$. $\boxed{y = -\cos(x) + C}$

159. Solve the ODE $y' = 5x + e^x$. (That is, find its general solution). $\boxed{y = \frac{5}{2}x^2 + e^x + C}$

160. Solve the ODE $x' = e^{3t}$. $\boxed{x = \frac{1}{3}e^{3t} + C}$

161. Solve the ODE $y' = \dfrac{x^4}{\sqrt{x^5+1}}$. $\boxed{y = \frac{2}{5} \cdot \sqrt{x^5+1} + C}$ See **Task 31d**.

An **initial value** (IV) or **initial condition** (IC) is a piece of information giving the value of a function or its derivative for a particular value of the input variable. An **initial value problem** (IVP) is a differential equation along with one or more initial values.

The general solution to an IVP ignores the initial condition and describes all solutions to the differential equation only.

162. Solve the IVP $y' = 4x^2 - 9,\ y(0) = 2$. $\boxed{y = \frac{4}{3}x^3 - 9x + 2}$

163. Solve the IVPs:

   (a) $x' = t,\ x(2) = 3$. This is **Task 150** with different letters. $\boxed{x = \frac{1}{2}t^2 + 1}$

   (b) $y' = t,\ y(2) = 3$. Again with different letters. $\boxed{y = \frac{1}{2}t^2 + 1}$

   (c) $y' = 5x + e^x,\ y(0) = 14$. $\boxed{y = \frac{5}{2}x^2 + e^x + 13}$

   (d) $y' = 5x + e^x,\ y(2) = 14$. $\boxed{y = \frac{5}{2}x^2 + e^x + 4 - e^2}$

164. Solve the IVP

$$\frac{\partial f}{\partial x} = \frac{6y}{x+1}, \quad \frac{\partial f}{\partial y} = 8e^y + 6\ln(x+1), \quad f(0,0) = 12.$$

From f'_x and f'_y we get $f(x,y) = 6y\ln(x+1) + 8e^y + C$. This would have $f(0,0) = 6 \cdot 0 \cdot 0 + 8 + C = 8 + C$, so in order for this to be 12 we need $C = 4$. Thus $\boxed{f(x,y) = 6y\ln(x+1) + 8e^y + 4}$.

165. The first-order ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y$$

has general solution

$$y = Ce^x.$$

Using this, give the solution to the IVP

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y, \quad y(0) = 5.$$

Setting $y(0) = Ce^0 = 5$ gives $C = 5$, so the soln is $\boxed{y(x) = 5e^x}$.

166. The first-order ODE

$$x' = t^2x^2$$

has general solution

$$x = \frac{-3}{t^3 + C}.$$

Using this, give the solution to the IVP

$$x' = t^2x^2, \quad x(1) = \tfrac{1}{2}.$$

Setting $x(1) = \dfrac{-3}{(1)^3 + C} = \dfrac{1}{2}$ gives $C = -7$ (this is **exactly Task 7**), so the solution is $\boxed{x(t) = \dfrac{-3}{t^3 - 7}}$.

167. The second-order ODE

$$y'' - 3y' - 10y = -40$$

has general solution

$$y = 4 + C_1e^{-2x} + C_2e^{5x}.$$

Using this, give the solution to the IVP

$$y'' - 3y' - 10y = -40, \qquad y(0) = 15, \quad y'(0) = -8.$$

$4 + C_1 + C_2 = 15$
$-2C_1 + 5C_2 = -8$
So $C_1 = 9$ and $C_2 = 2$.
Solution to the IVP: $\boxed{y = 4 + 9e^{-2x} + 2e^{5x}}$