Analysis 2, Summer 2024

List 8

First-order linear ODEs

187. Evaluate the following integrals using integration by parts:

 (a) $\int x\sin(x)\,\mathrm{d}x = \boxed{\sin(x) - x\cos(x) + C}$

 (b) $\int 3x\ln(x)\,\mathrm{d}x = \boxed{\frac{3}{2}x^2\ln(x) - \frac{3}{4}x^2}$

 (c) $\int e^{5t/2}\sin(5t)\,\mathrm{d}t = \boxed{\frac{2}{25}e^{5t/2}\sin(5t) - \frac{4}{25}e^{5t/2}\cos(5t)}$

An order n **linear** ODE for y can be written in the form

$$a_n\,y^{(n)} + \cdots + a_2\,y'' + a_1\,y' + a_0\,y = f,$$

where $a_i = a_i(t)$ and $f = f(t)$ are functions. The equation is called **homogeneous** if $f(t) = 0$ and **non-homogeneous** otherwise. If every $a_i(t)$ is a constant function, the equation **has constant coefficients** (note $f(t)$ can still be non-constant).

188. For each ODE below, state whether it is linear or not:

 (a) $y\cdot y' = 5t+2$ $\boxed{\text{not linear}}$

 (b) $y' + y = t^2$ $\boxed{\text{linear}}$

 (c) $y' + y^2 = t^2$ $\boxed{\text{not linear}}$

 (d) $y' + ty = t^3$ $\boxed{\text{linear}}$

 (e) $\frac{y''}{t^2} = 3e^t + y$ $\boxed{\text{linear}}$ (can be re-written as $y'' - t^2y = 3t^2\sin(t)$)

 (f) $y = y' - e^t$ $\boxed{\text{linear}}$

 (g) $y' = e^y + t$ $\boxed{\text{not linear}}$

189. What is the order of the linear ODE $t^4y'' + t^3y' - y = t^5$? order $\boxed{2}$ because of y''.

190. The following ODEs are linear. Label each as homogeneous or non-homogeneous.

 (a) $y' + (3t^2 - 6t + 14)y = 0$ $\boxed{\text{homogeneous}}$

 (b) $2x' + 3x = 4$ $\boxed{\text{non-homogeneous}}$

 (c) $x' + (3t-1)x = t^2x$ $\boxed{\text{homogeneous}}$ (can be re-written as $x' + (3t-1-t^2)x = 0$)

 (d) $y'' = 9y$ $\boxed{\text{homogeneous}}$

 (e) $y'' + 9y' + t = 0$ $\boxed{\text{non-homogeneous}}$ ($y'' + 9y' + 0y = -t$)

191. "Every homogeneous first-order linear ODE is autonomous." Either use formulas to explain why this is true *or* give an example that shows this claim is false. $\boxed{\text{False}}$. One example: $y' + y = 1$ is not autonomous.

192. "Every homogeneous first-order linear ODE is separable." Either use formulas to explain why this is true *or* give an example that shows this claim is false. $\boxed{\text{True}}$: $y' + a(x)y = 0$ is also $y' = -a(x)y$, which is exactly $y' = h(x)g(y)$, and therefore separable, with $h(x) = -a(x)$ and $g(y) = y$.

> The **homogeneous first-order linear ODE** $y' = -a(x) \cdot y$ has general solution
> $$y = Ce^{-A(x)},$$
> where $A(x)$ is any antiderivative of $a(x)$, meaning $A'(x) = a(x)$.
> For the **non-homogeneous first-order linear ODE** $y' + a(x) \cdot y = f(x)$, the general solution is
> $$y = \left(\int e^{A(x)} \cdot f(x)\,\mathrm{d}x\right) e^{-A(x)}.$$
> This can also be written as $y = \frac{1}{M(x)} \int M(x) f(x)\,\mathrm{d}x$, where $M(x) = e^{A(x)}$ is called the "integrating factor".

193. Solve the following differential equations:

(a) $y' - 2y = 7$. Method 1: Multiply the whole thing by e^{-2x} to get

$$\begin{aligned} e^{-2x}y' - 2e^{-2x}y &= 7e^{-2x} \\ \left(e^{-2x}y\right)' &= 7e^{-2x} \\ e^{-2x}y &= \tfrac{-7}{2}e^{-2x} + C \\ y &= \boxed{\tfrac{-7}{2} + Ce^{2x}} \end{aligned}$$

Method 2: Use the formula from the box to get

$$y' = \left(\int e^{-2x} \cdot 7\,\mathrm{d}x\right)e^{2x} = \left(\frac{-7}{2}e^{-2x} + C\right)e^{2x} = \boxed{\tfrac{-7}{2} + Ce^{2x}}.$$

(b) $y' = y + 2x - 3$.
This is of the form $y' + ay = b$ with $a(x) = -1$ and $b(x) = 2x - 3$. Let $A(x) = -x$. Then

$$y' = e^{x}\left(\int e^{-x}(2x-3)\,\mathrm{d}x\right) = e^{x}\left((1-2x)e^{-x} + C\right) = \boxed{Ce^{x} - 2x + 1}.$$

(c) $y' + (\sin t)y = e^{\cos t}$. $\boxed{y = e^{\cos t}(t + C)}$

(d) $y' - \left(\dfrac{3x^2 - 2x + 3}{x^2 + 1}\right)y - \dfrac{e^{3x}}{x^2+1} = 0$. $a(x) = \frac{3x^2-2x+3}{x^2+1} = 3 - \frac{2x}{x^2+1}$.
$A(x) = \ln(x^2+1) - 3x = \ln\left(\dfrac{x^2+1}{e^{3x}}\right)$ leads to $\boxed{y = \dfrac{e^{3x}}{x^2+1}(x + C)}$.

194. Solve the following initial value problems:

(a) $y' = y + 2x - 3$, $\quad y(0) = 9$. From **Task 193b**, $y = Ce^{x} - 2x + 1$ for some C. For the IVP, $9 = Ce^{0} - 2(0) + 1 = C + 1$, so $\boxed{y = 8e^{x} - 2x + 1}$.

(b) $y' = \dfrac{-xy}{x+1}$, $\quad y(1) = 3$. From $\displaystyle\int \frac{\mathrm{d}y}{y} = \int \frac{-x}{x+1}\,\mathrm{d}x$ we get $\ln(y) = \ln(x+1) - x + C$, and so the general solution is $y = Ce^{-x}(x+1)$. From $y(1) = 3$ we get $C = \frac{3e}{2}$ and so $\boxed{y = \tfrac{3}{2}e^{1-x}(x+1)}$.

(c) $\dfrac{y'}{\cos(t)} + y\tan(t) = \dfrac{e^{\cos t}}{\cos(t)}$, $\quad y(0) = e$. After multiplying this entire ODE by $\cos(t)$, it becomes exactly the ODE from **Task 193c**. The general solution is thus $y = e^{\cos t}(t+C)$, and $y(0) = e$ implies $C = 1$, so $\boxed{y = e^{\cos t}(t+1)}$.

(d) $x^2 y' = y^2$, $\quad y(2) = 4$. General: $y = \dfrac{x}{Cx+1}$. Initial condition leads to $C = \frac{1}{4}$ and $\boxed{y = \dfrac{4x}{4-x}}$.

195. Solve the initial value problem

$$2y' - y = 4\sin(3t), \quad y(0) = y_0,$$

and then answer the following questions:

(a) For which values of y_0 does $y(t)$ go towards $+\infty$ as $t \to +\infty$?

(b) For which values of y_0 does $y(t)$ go towards $-\infty$ as $t \to +\infty$?

(c) For which values of y_0 does $y(t)$ remain bounded as $t \to +\infty$?

This is a linear first-order ODE with general solution

$$y = \tfrac{-4}{37}\big(\sin(3t) + 6\cos(3t)\big) + Ce^{t/2}.$$

As $t \to \infty$, the trig part $\frac{-4}{37}(\dots)$ is bounded but the exponential part $Ce^{t/2}$ can go to $+\infty$, $-\infty$, or 0 depending on whether C is positive, negative, or 0. Plugging in $t = 0$ gives $y(0) = \frac{-24}{37} + C$, and this must equal y_0, so $C = \frac{24}{37} + y_0$.

a) $\lim\limits_{t\to\infty} y(t) = \infty$ when $C > 0$, which is when $\frac{24}{37} + y_0 > 0$, or $\boxed{y_0 > \dfrac{-24}{37}}$.

b) When $C < 0$, which is $\boxed{y_0 < \dfrac{-24}{37}}$.

b) When $C = 0$, which is $\boxed{y_0 = \dfrac{-24}{37}}$.

196. Solve the non-homogeneous first-order linear ODE

$$y' - \tan(t)\,y = 2t\sec(t) \qquad (*)$$

in three different ways:

(a) "Variation of parameters." The solution to the homogeneous equation

$$y' - \tan(t)\,y = 0$$

is $y = C\sec(t)$ for a constant number C. Assume that the solution to $(*)$ is of the form

$$y = g(t)\cdot\sec(t)$$

for some function $g(t)$, and determine what $g(t)$ must be.

$y' = g(t)\sec(t)\tan(t) + g'(t)\sec(t)$, so

$$\begin{aligned} y' - \tan(t)\,y &= g(t)\sec(t)\tan(t) + g'(t)\sec(t) - \tan(t)g(t)\sec(t) \\ y' - \tan(t)\,y &= g'(t)\sec(t) \\ 2t\sec(t) &= g'(t)\sec(t) \end{aligned}$$

gives that $g'(t) = 2t$, so $g(t) = t^2 + C$, and thus $\boxed{y = (t^2 + C)\sec(t)}$.

(b) "Integrating factor." Multiplying $(*)$ by any function $M(t)$ gives

$$My' - M\tan(t)\,y = M\,2t\sec(t).$$

If $M(t)\tan(t)$ were exactly $-M'(t)$, then we could re-write this as

$$My' + M'y = M\,2t\sec(t)$$
$$\big(My\big)' = M\,2t\sec(t).$$

The solution to $M\tan(t) = -M'$ is $M = e^{(-\int \tan(t)\,\mathrm{d}t)}$. Use this to solve $(*)$.

$y = \frac{1}{\cos(t)}\displaystyle\int \cos(t)2t\sec(t)\,\mathrm{d}t = \sec(t)\int 2t\,\mathrm{d}t = \boxed{(t^2+C)\sec(t)}$.

(c) Big formula. $y' + a(t)y = f(t)$ is always solved by $y = \left(\displaystyle\int e^{A(t)}f(t)\,\mathrm{d}t\right)e^{-A(t)}$, where $A'(t) = a(t)$. Use $a(t) = -\tan(t)$ and $f(t) = 2t\sec(t)$ in this formula to solve $(*)$.

$A(t) = -\log(\cos(t))$, so $y = \left(\displaystyle\int e^{\log(\cos(t))}2t\sec(t)\,\mathrm{d}t\right)e^{-\log(\cos(t))}$

$= \left(\displaystyle\int 2t\,\mathrm{d}t\right)\sec(t) = \boxed{(t^2+C)\sec(t)}$.

197. Solve the ODE

$$t\,y' + y = t^3$$

using any of the three methods from the previous task.

Standard form: $y' + \frac{1}{t}y = t^2$. General solution: $\boxed{y = \frac{t^3}{4} + \frac{C}{t}}$.

The **Laplace transform** of a function $f = f(t)$ is written $\mathscr{L}\big[f\big]$ and is the function

$$F(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t = \lim_{b\to\infty}\int_0^b f(t)e^{-st}\,\mathrm{d}t.$$

Note that $\mathscr{L}\big[f\big]$ is a function of "s" while f was a function of "t" (these are the most common letters to use; what is important is that they are not the same variable). Instead of computing integrals every time, we often use these common examples:

$f(t)$	t^n	e^{rt}	$\sin(\omega t)$	$\cos(\omega t)$
$F(t)$	$\frac{n!}{s^{n+1}}$	$\frac{1}{s-r}$	$\frac{\omega}{s^2+\omega^2}$	$\frac{s}{s^2+\omega^2}$

☆198. Find $\mathscr{L}\big[t^2\big]$ by computing $\displaystyle\int_0^\infty t^2e^{-st}\,\mathrm{d}t$. This requires integration by parts twice.

Final answer: $\boxed{\frac{2}{s^3}}$

199. For each of the functions below, find $F(s) = \mathscr{L}\big[f(t)\big]$ using the table of common Laplace transforms.

(a) $f(t) = 1$. $\mathscr{L}\big[t^0\big] = \frac{0!}{s^{0+1}} = \boxed{\frac{1}{s}}$ or $\mathscr{L}\big[e^{0t}\big] = \frac{1}{s-0} = \boxed{\frac{1}{s}}$

(b) $f(t) = t$. $\boxed{\frac{1}{s^2}}$

(c) $f(t) = t^5$. $\boxed{\frac{120}{s^6}}$

(d) $f(t) = e^t$. $\boxed{\frac{1}{s-1}}$

(e) $f(t) = e^{-t}$. $\boxed{\frac{1}{s+1}}$

(f) $f(t) = \sin(9t)$. $\boxed{\frac{9}{s^2+81}}$

200. Give $\mathscr{L}^{-1}\left[\frac{-2}{s^2+4}\right]$, the inverse Laplace transform of $\frac{-2}{s^2+4}$. That is, find a function $f(t)$ for which $F(s) = \mathscr{L}\big[f(t)\big] = \frac{-2}{s^2+4}$. $\sin(-2t) = \boxed{-\sin(2t)}$

201. Solve the equation

$$sY - 4 - 3Y = \frac{1}{s-5}$$

for Y. (This is only an algebra task. It could have been on List 0.) $\boxed{Y = \frac{4s-19}{s^2-8s+15}}$

202. Find numbers A and B such that $\frac{5s-28}{s^2-10s+16} = \frac{A}{s-2} + \frac{B}{s-8}$.

$$\frac{5s-28}{(s-2)(s-8)} = \frac{A}{s-2} + \frac{B}{s-8}$$
$$5s - 28 = A(s-8) + B(s-2)$$
$$5s - 28 = As - 8A + Bs - 2B$$
$$5s - 28 = (A+B)s + (-8A - 2B)$$

Therefore $\begin{cases} A + B = 5 \\ -8A - 2B = -28 \end{cases}$. The solution to this system is $\boxed{A = 3, B = 2}$.

203. Solve

$$6F(s) + s \cdot F(s) - 4 = \frac{8}{s}$$

for $F(s)$ and then give the partial fraction decomposition of $F(s)$.

$\boxed{F(s) = \frac{-4s+8}{s^2+6s} = \frac{4/3}{s} + \frac{-16/3}{s+6}}$

Properties:

- $\mathscr{L}\big[c \cdot f(t) + g(t)\big] = c\mathscr{L}\big[f(t)\big] + \mathscr{L}\big[g(t)\big]$ for constant c.
- $\mathscr{L}\big[f(ct)\big] = \frac{1}{c}F(\frac{s}{c})$ for constant c.
- $\mathscr{L}\big[e^{kt} \cdot f(t)\big] = F(s-k)$ for constant k.
- $\mathscr{L}\big[t \cdot f(t)\big] = -\frac{\mathrm{d}F}{\mathrm{d}s}$. This implies $\mathscr{L}\big[t^n f\big] = (-1)^n \cdot F^{(n)}(s)$.
- $\mathscr{L}\big[f'(t)\big] = s \cdot F(s) - f(0)$. This implies $\mathscr{L}\big[f''\big] = s^2 \cdot F(s) - s \cdot f(0) - f'(0)$.

204. For the functions

$$x(t) = te^t, \qquad y(t) = e^{5t}\sin(2t), \qquad z(t) = 2t^2 + e^{5t}t^3,$$

find $X(s) = \mathscr{L}[x(t)]$ and $Y(s) = \mathscr{L}[y(t)]$ and $Z(s) = \mathscr{L}[z(t)]$.

For x, Method 1: $\int_0^\infty te^te^{-st}\,\mathrm{d}t = \int_0^\infty te^{(1-s)t}\,\mathrm{d}t = \boxed{\frac{1}{(s-1)^2}}$ using integration by parts.

Method 2: Set $f(t) = t$, so $F(s) = \frac{1}{s^2}$. Then $\mathscr{L}[e^{1t}f(t)] = F(s-1) = \boxed{\frac{1}{(s-1)^2}}$.

Method 3: Set $g(t) = e^t$, so $G(s) = \frac{1}{s-1}$. Then $\mathscr{L}[t\,g(t)] = -G'(s) = \boxed{\frac{1}{(s-1)^2}}$

$\boxed{Y(s) = \frac{2}{(s-5)^2+4}}$ $\qquad$ $\boxed{Z(s) = \frac{4}{s^3} + \frac{6}{(s-5)^4}}$

205. Find the function from its Laplace transform:

(a) $F(s) = \frac{1}{s^2+16}$. $\boxed{\frac{1}{4}\sin(4t)}$

(b) $F(s) = \frac{9}{s^2+3s}$. $F(s) = \frac{9}{s(s+3)} = \frac{3}{s} + \frac{-3}{s+3}$, so $f(t) = \boxed{3 - 3e^{-3t}}$.

(c) $X(s) = \frac{s-4}{s^2-4}$. $F(s) = \frac{s-4}{(s+2)(s-2)} = \frac{3/2}{s+2} + \frac{-1/2}{s-2}$, so $f(t) = \boxed{\frac{3}{2}e^{-2t} - \frac{1}{2}e^{2t}}$.

(d) $Y(s) = \frac{7s+6}{s^2+9}$. $F(s) = 7\left(\frac{s}{s^2+9}\right) + 2\left(\frac{3}{s^2+9}\right)$, so $f(t) = \boxed{7\cos(3t) + 2\sin(3t)}$.

(e) $F(s) = \frac{5s-28}{s^2-10s+16}$. $F(s) = \frac{3}{s-2} + \frac{2}{s-8}$ from **Task 202**, so $f(t) = \boxed{3e^{2t} + 2e^{8t}}$.

206. Re-write $\mathscr{L}[x'] = \mathscr{L}[3x - 7y]$ as an equation with X, Y, s, and the number $x(0)$. Here $X = X(s)$ is the Laplace transform of $x = x(t)$, and similarly $Y = \mathscr{L}[y]$.

$\boxed{sX - x(0) = 3X - 7Y}$

207. Solve the non-homogeneous first-order linear IVP

$$y' - 3y = e^{5t}, \qquad y(0) = 4$$

in four different ways:

(a) "Variation of parameters"

(b) "Integrating factor"

(c) "Big formula"

(d) "Laplace transformation"

$$\mathscr{L}[y'] - 3\mathscr{L}[y] = \mathscr{L}[e^{5t}]$$
$$(sY - 4) - 3Y = \frac{1}{s-5}$$

Solving this last equation for Y (see **Task 201**) gives

$$\begin{aligned} Y &= \frac{4s-19}{s^2-8s+15} \\ y &= \mathscr{L}^{-1}\left[\frac{4s-19}{s^2-8s+15}\right] \\ y &= \mathscr{L}^{-1}\left[\frac{7/2}{s-3}+\frac{1/2}{s-5}\right] \\ y &= \frac{7}{2}\mathscr{L}^{-1}\left[\frac{1}{s-3}\right]+\frac{1}{2}\mathscr{L}^{-1}\left[\frac{1}{s-5}\right] \\ &\boxed{y = \frac{7}{2}e^{3t}+\frac{1}{2}e^{5t}} \end{aligned}$$

208. Solve the IVP

$$x' + 6x = 8, \qquad x(0) = -4$$

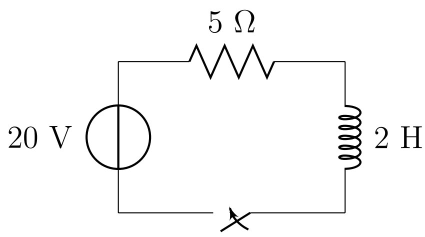
using any of the four methods from the previous task.

Laplace: $(sX-4)+6X = \dfrac{8}{s}$. From **Task 203**,

$$X = \frac{-4s+8}{s^2+6s} = \frac{4/3}{s} + \frac{-16/3}{s+6}$$

so $\boxed{x = \frac{4}{3} - \frac{16}{3}e^{-6t}}$.

209. *RL circuit (DC):* When the switch in the circuit



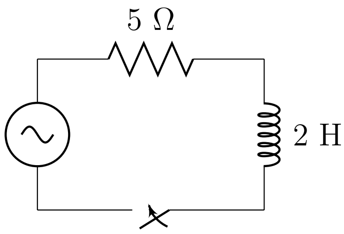
is closed, the current $i = i(t)$ flowing through the circuit satisfies

$$2\frac{\mathrm{d}i}{\mathrm{d}t} + 5i = 20, \quad i(0) = 0.$$

Solve this IVP.

Note: this task is **not** starred. I will never require you to analyze a circuit in this class, but I do expect you to be able to solve $2y' + 5y = 20$, $y(0) = 0$, which is exactly this task using different letters. $\boxed{i(t) = 4 - 4e^{-2.5t}}$

210. *RL circuit (AC):* If the direct-current battery in Task 209 is replaced by an alternating current source $\mathcal{E} = 20\sin(5t)$,



the differential equation becomes

$$2\frac{\mathrm{d}i}{\mathrm{d}t} + 5i = 20\sin(5t), \quad i(0) = 0.$$

Solve this IVP.

Int. factor: $M = e^{\int \frac{5}{2}\,\mathrm{d}t} = e^{2.5t}$, and (see **Task 187c**)

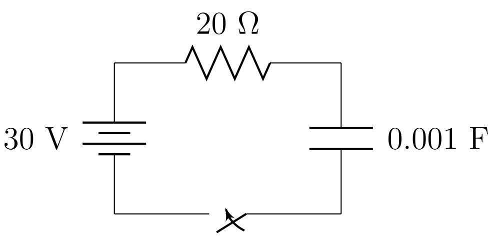
$$\int Mf\,\mathrm{d}t = 20\int e^{2.5t}\sin(5t)\,\mathrm{d}t = 20\left(\frac{2}{25}e^{2.5t}\sin(5t) - \frac{4}{25}e^{2.5t}\cos(5t)\right)$$

Laplace: $5I - 2sI = 20(\frac{5}{s^2+25})$, so

$$I = \frac{100}{(2s+5)(s^2+25)} = \frac{16}{10s+25} + \frac{4}{s^2+25} - \frac{8\,s}{5s^2+125}$$

Either way, $\boxed{i = \frac{8}{5}e^{-2.5t} + \frac{4}{5}\sin(5t) - \frac{8}{5}\cos(5t)}$

211. *RC circuit, charging:* When the switch in the circuit

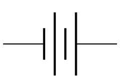
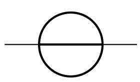
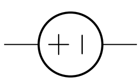


is closed, the charge $q = q(t)$ in the capacitor satisfies

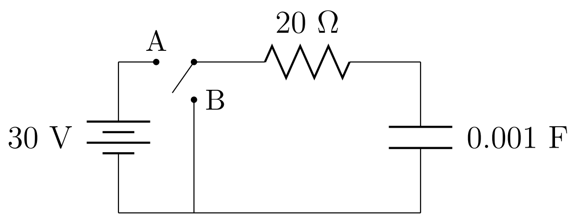
$$20\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{0.001}q = 30, \quad q(0) = 0.$$

Solve the IVP, and then determine $\lim_{t\to\infty} q(t)$.

$\boxed{q = 0.03(1 - e^{-50t})}$, so as $t \to \infty$ we get $\boxed{q \to 0.03}$, or 30 mC.

Note:  and  and  are basically all the same.

212. *RC circuit, discharging:* When the switch in the circuit



is at position A, the circuit behaves like the one in Task 211. When the switch moves to position B at $t = 0$, the capacitor starts discharging according to

$$20\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{0.001}q = 0, \quad q(0) = 0.03$$

Solve this IVP. $\boxed{q = 0.03e^{-50t}}$

☆213. For each of the following, your formula for $q(t)$ will have some or all of $R, C, \mathcal{E}, Q_0$ as constants.

(a) Charging from zero: Solve $q' + \frac{1}{RC}q = \frac{\mathcal{E}}{R}, \quad q(0) = 0.$ $\boxed{q(t) = C\mathcal{E}(1 - e^{-t/(RC)})}$

(b) Discharging: Solve $q' + \frac{1}{RC}q = 0, \quad q(0) = Q_0.$ $\boxed{q(t) = Q_0 e^{-t/(RC)}}$

(c) Charging from a non-zero start: Solve $q' + \frac{1}{RC}q = \frac{\mathcal{E}}{R}, \quad q(0) = Q_0.$
$\boxed{q(t) = C\mathcal{E} + (Q_0 - C\mathcal{E})e^{-t/(RC)}}$

(d) Compare part (c) to Task 182, Newton's Law of Cooling. They are basically the same, just with different names for the constants!

$$\text{function}(t) = a + be^{-ct}$$