Analysis 2, Summer 2024

List 8

First-order linear ODEs

187. Evaluate the following integrals using integration by parts:

(a) $\displaystyle\int x\sin(x)\,\mathrm{d}x$ (b) $\displaystyle\int 3x\ln(x)\,\mathrm{d}x$ (c) $\displaystyle\int e^{5t/2}\sin(5t)\,\mathrm{d}t$

An order n **linear** ODE for y can be written in the form

$$a_n\,y^{(n)} + \cdots + a_2\,y'' + a_1\,y' + a_0\,y = f,$$

where $a_i = a_i(t)$ and $f = f(t)$ are functions. The equation is called **homogeneous** if $f(t) = 0$ and **non-homogeneous** otherwise. If every $a_i(t)$ is a constant function, the equation **has constant coefficients** (note $f(t)$ can still be non-constant).

188. For each ODE below, state whether it is linear or not:

(a) $y \cdot y' = 5t + 2$

(b) $y' + y = t^2$

(c) $y' + y^2 = t^2$

(d) $y' + ty = t^3$

(e) $\dfrac{y''}{t^2} = 3e^t + y$

(f) $y = y' - e^t$

(g) $y' = e^y + t$

189. What is the order of the linear ODE $t^4y'' + t^3y' - y = t^5$?

190. The following ODEs are linear. Label each as homogeneous or non-homogeneous.

(a) $y' + (3t^2 - 6t + 14)y = 0$

(b) $2x' + 3x = 4$

(c) $x' + (3t - 1)x = t^2x$

(d) $y'' = 9y$

(e) $y'' + 9y' + t = 0$

191. "Every homogeneous first-order linear ODE is autonomous." Either use formulas to explain why this is true *or* give an example that shows this claim is false.

192. "Every homogeneous first-order linear ODE is separable." Either use formulas to explain why this is true *or* give an example that shows this claim is false.

The **homogeneous first-order linear ODE** $y' = -a(x)\cdot y$ has general solution

$$y = Ce^{-A(x)},$$

where $A(x)$ is any antiderivative of $a(x)$, meaning $A'(x) = a(x)$.

For the **non-homogeneous first-order linear ODE** $y' + a(x)\cdot y = f(x)$, the general solution is

$$y = \left(\int e^{A(x)} \cdot f(x)\,\mathrm{d}x\right) e^{-A(x)}.$$

This can also be written as $y = \frac{1}{M(x)}\int M(x)f(x)\,\mathrm{d}x$, where $M(x) = e^{A(x)}$ is called the "integrating factor".

193. Solve the following differential equations:

(a) $y' - 2y = 7$. (b) $y' = y + 2x - 3$. (c) $y' + (\sin t)y = e^{\cos t}$.

(d) $y' - \left(\dfrac{3x^2 - 2x + 3}{x^2 + 1}\right) y - \dfrac{e^{3x}}{x^2 + 1} = 0$.

194. Solve the following initial value problems:

(a) $y' = y + 2x - 3, \quad y(0) = 9$.

(b) $y' = \dfrac{-xy}{x+1}, \quad y(1) = 3$.

(c) $\dfrac{y'}{\cos(t)} + y\tan(t) = \dfrac{e^{\cos t}}{\cos(t)}, \quad y(0) = e$.

(d) $x^2\, y' = y^2, \quad y(2) = 4$.

195. Solve the initial value problem

$$2y' - y = 4\sin(3t), \quad y(0) = y_0,$$

and then answer the following questions:

(a) For which values of y_0 does $y(t)$ go towards $+\infty$ as $t \to +\infty$?

(b) For which values of y_0 does $y(t)$ go towards $-\infty$ as $t \to +\infty$?

(c) For which values of y_0 does $y(t)$ remain bounded as $t \to +\infty$?

196. Solve the non-homogeneous first-order linear ODE

$$y' - \tan(t)\, y = 2t\sec(t) \qquad (*)$$

in three different ways:

(a) "Variation of parameters." The solution to the homogeneous equation

$$y' - \tan(t)\, y = 0$$

is $y = C\sec(t)$ for a constant number C. Assume that the solution to $(*)$ is of the form

$$y = g(t) \cdot \sec(t)$$

for some function $g(t)$, and determine what $g(t)$ must be.

(b) "Integrating factor." Multiplying $(*)$ by any function $M(t)$ gives

$$My' - M\tan(t)\, y = M\, 2t\sec(t).$$

If $M(t)\tan(t)$ were exactly $-M'(t)$, then we could re-write this as

$$My' + M'y = M\, 2t\sec(t)$$
$$\big(My\big)' = M\, 2t\sec(t).$$

The solution to $M\tan(t) = -M'$ is $M = e^{(-\int \tan(t)\,\mathrm{d}t)}$. Use this to solve $(*)$.

(c) Direct formula. $y' + a(t)y = f(t)$ is always solved by $y = \left(\displaystyle\int e^{A(t)} f(t)\,\mathrm{d}t\right) e^{-A(t)}$, where $A'(t) = a(t)$. Use $a(t) = -\tan(t)$ and $f(t) = 2t\sec(t)$ in this formula to solve $(*)$.

197. Solve the ODE

$$t\,y' + y = t^3$$

using any of the three methods from the previous task.

The **Laplace transform** of a function $f = f(t)$ is written $\mathscr{L}\big[f\big]$ and is the function

$$F(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t = \lim_{b\to\infty}\int_0^b f(t)e^{-st}\,\mathrm{d}t.$$

Note that $\mathscr{L}\big[f\big]$ is a function of "s" while f was a function of "t" (these are the most common letters to use; what is important is that they are not the same variable). Instead of computing integrals every time, we often use these common examples:

$f(t)$	t^n	e^{rt}	$\sin(\omega t)$	$\cos(\omega t)$
$F(t)$	$\dfrac{n!}{s^{n+1}}$	$\dfrac{1}{s-r}$	$\dfrac{\omega}{s^2+\omega^2}$	$\dfrac{s}{s^2+\omega^2}$

☆198. Find $\mathscr{L}\big[t^2\big]$ by computing $\displaystyle\int_0^\infty t^2 e^{-st}\,\mathrm{d}t.$

199. For each of the functions below, find $F(s) = \mathscr{L}\big[f(t)\big]$ using the table of common Laplace transforms.

(a) $f(t) = 1.$
(b) $f(t) = t.$
(c) $f(t) = t^5.$
(d) $f(t) = e^t.$
(e) $f(t) = e^{-t}.$
(f) $f(t) = \sin(9t).$

200. Give $\mathscr{L}^{-1}\left[\dfrac{-2}{s^2+4}\right]$, the inverse Laplace transform of $\dfrac{-2}{s^2+4}$. That is, find a function $f(t)$ for which $F(s) = \mathscr{L}\big[f(t)\big] = \dfrac{-2}{s^2+4}$.

201. Solve the equation

$$sY - 4 - 3Y = \frac{1}{s-5}$$

for Y. (This is only an algebra task. It could have been on List 0.)

202. Find numbers A and B such that $\dfrac{5s-28}{s^2-10s+16} = \dfrac{A}{s-2} + \dfrac{B}{s-8}$.

203. Solve

$$6F(s) + s\cdot F(s) - 4 = \frac{8}{s}$$

for $F(s)$ and then give the partial fraction decomposition of $F(s)$.

Properties:

- $\mathscr{L}\big[c\cdot f(t) + g(t)\big] = c\mathscr{L}\big[f(t)\big] + \mathscr{L}\big[g(t)\big]$ for constant c.
- $\mathscr{L}\big[f(ct)\big] = \frac{1}{c}F(\frac{s}{c})$ for constant c.
- $\mathscr{L}\big[e^{kt}\cdot f(t)\big] = F(s-k)$ for constant k.
- $\mathscr{L}\big[t\cdot f(t)\big] = -\frac{\mathrm{d}F}{\mathrm{d}s}$. This implies $\mathscr{L}\big[t^n f\big] = (-1)^n\cdot F^{(n)}(s)$.
- $\mathscr{L}\big[f'(t)\big] = s\cdot F(s) - f(0)$. This implies $\mathscr{L}\big[f''\big] = s^2\cdot F(s) - s\cdot f(0) - f'(0)$.

204. For the functions

$$x(t) = te^t, \qquad y(t) = e^{5t}\sin(2t), \qquad z(t) = 2t^2 + e^{5t}t^3,$$

find $X(s) = \mathscr{L}\big[x(t)\big]$ and $Y(s) = \mathscr{L}\big[y(t)\big]$ and $Z(s) = \mathscr{L}\big[z(t)\big]$.

205. Find the function from its Laplace transform:

(a) $F(s) = \dfrac{1}{s^2+16}$.

(b) $F(s) = \dfrac{9}{s^2+3s}$. Hint: first re-write F as $\frac{?}{s} + \frac{?}{s+3}$.

(c) $X(s) = \dfrac{s-4}{s^2-4}$.

(d) $Y(s) = \dfrac{7s+6}{s^2+9}$.

(e) $F(s) = \dfrac{5s-28}{s^2-10s+16}$.

206. Re-write $\mathscr{L}\big[x'\big] = \mathscr{L}\big[3x - 7y\big]$ as an equation with X, Y, s, and the number $x(0)$.
Here $X = X(s)$ is the Laplace transform of $x = x(t)$, and similarly $Y = \mathscr{L}\big[y\big]$.

207. Solve the non-homogeneous first-order linear IVP

$$y' - 3y = e^{5t}, \qquad y(0) = 4$$

in four different ways:

(a) “Variation of parameters”

(b) “Integrating factor”

(c) “Direct formula”

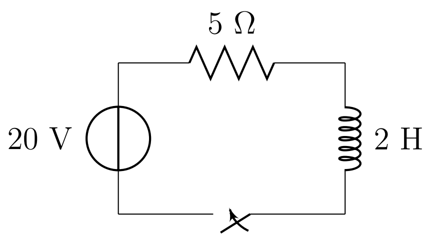
(d) “Laplace transformation”

208. Solve the IVP

$$x' + 6x = 8, \qquad x(0) = -4$$

using any of the four methods from the previous task.

209. *RL circuit (DC):* When the switch in the circuit



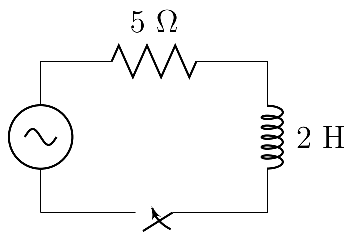
is closed, the current $i = i(t)$ flowing through the circuit satisfies

$$2\frac{\mathrm{d}i}{\mathrm{d}t} + 5i = 20, \quad i(0) = 0.$$

Solve this IVP.

Note: this task is **not** starred. I will never require you to analyze a circuit in this class, but I do expect you to be able to solve $2y' + 5y = 20,\ y(0) = 0$, which is exactly this task using different letters.

210. *RL circuit (AC):* If the direct-current battery in Task 209 is replaced by an alternating current source $\mathcal{E} = 20\sin(5t)$,

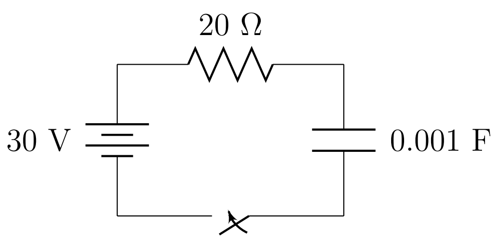


the differential equation becomes

$$2\frac{\mathrm{d}i}{\mathrm{d}t} + 5i = 20\sin(5t), \quad i(0) = 0.$$

Solve this IVP.

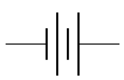
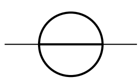
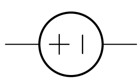
211. *RC circuit, charging:* When the switch in the circuit



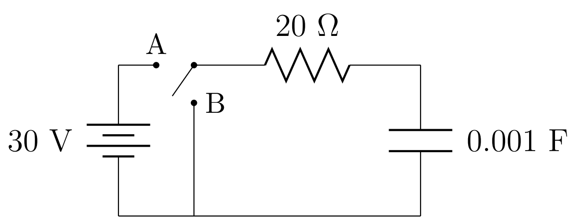
is closed, the charge $q = q(t)$ in the capacitor satisfies

$$20\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{0.001}q = 30, \quad q(0) = 0.$$

Solve the IVP, and then determine $\lim_{t\to\infty} q(t)$.

Note:  and  and  are basically all the same.

212. *RC circuit, discharging:* When the switch in the circuit



is at position A, the circuit behaves like the one in Task 211. When the switch moves to position B at $t = 0$, the capacitor starts discharging according to

$$20\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{0.001}q = 0, \quad q(0) = 0.03$$

Solve this IVP.

☆213. For each of the following, your formula for $q(t)$ will have some or all of $R, C, \mathcal{E}, Q_0$ as constants.

(a) Charging from zero: Solve $q' + \frac{1}{RC}q = \frac{\mathcal{E}}{R}, \quad q(0) = 0.$

(b) Discharging: Solve $q' + \frac{1}{RC}q = 0, \quad q(0) = Q_0.$

(c) Charging from a non-zero start: Solve $q' + \frac{1}{RC}q = \frac{\mathcal{E}}{R}, \quad q(0) = Q_0.$

(d) Compare part (c) to Task 60.