Analysis 2, Summer 2024

List 9

Higher-order linear ODEs

☆214. Solve the ODE $t^2y'' + 3ty' + y = 0$.

Note: This task is starred. I would *not* ask you to solve this ODE on any quiz or exam because it is second-order but does *not* have constant coefficients.

It is not required for this course, but if you are curious here is a method that can solve this ODE.

Substitute $z = t \cdot y$, so $y = \frac{z}{t} = t^{-1} \cdot z$. Then

$$\begin{aligned} t^2y'' + 3ty' + y &= t^2(t^{-1} \cdot z)'' + 3t(t^{-1} \cdot z)' + (t^{-1} \cdot z) \\ &= t^2(t^{-1}z' - t^{-2}z)' + 3t(t^{-1}z' - t^{-2}z) + t^{-1}z \\ &= t^2((t^{-1}z'' - t^{-2}z') - (t^{-2}z' - 2t^{-3}z)) + 3z' - 3t^{-1}z + t^{-1}z \\ &= 0(tz'' - z' - z' + 2t^{-1}z) + 3z' - 2t^{-1}z \\ &= tz'' + z' \end{aligned}$$

So the ODE becomes $tz'' + z' = 0$. Substituting $u = z'$, so $u' = z''$, this is

$$\begin{aligned} tz'' + z' &= 0 \\ tu' + u &= 0 \qquad (\textit{separable}) \\ t\frac{\mathrm{d}u}{\mathrm{d}t} &= -u \\ \int \frac{1}{u}\mathrm{d}u &= \int \frac{-1}{t}\mathrm{d}t \\ \ln(u) &= -\ln(t) + C \\ u &= Ce^{-\ln(t)} = \frac{C}{t} \end{aligned}$$

Then $z' = \frac{C_1}{t}$ means $z = \int \frac{C_1}{t}\mathrm{d}t = C_1 \ln(t) + C_2$. Finally, $y = \frac{z}{t}$, so

$$\boxed{y = \frac{C_1}{t} + \frac{C_2 \ln t}{t}}$$

215. The second-order ODE

$$t^2y'' + 3ty' + y = 0$$

has general solution

$$y = \frac{C_1}{t} + \frac{C_2 \ln t}{t}.$$

(a) Use this general solution to find a formula for y'. $\boxed{y' = -\frac{C_1}{t^2} + \frac{C_2}{t^2} - \frac{C_2 \ln(t)}{t^2}}$

(b) Using the formulas for y and y', give the solution to the IVP

$$t^2y'' + 3ty' + y = 0, \quad y(1) = 2, \quad y'(1) = 5.$$

Note: This is the same ODE has Task 214, but this task is not starred.

From $y(1) = 2$ we get $\frac{C_1}{1} + 0 = 2$, so $C_1 = 2$. Using y' from part (a), we have $y'(1) = -C_1 + C_2$, so the second initial condition gives $-(2) + C_2 = 5$, or $C_2 = 7$.

The solution to the IVP is therefore $\boxed{y = \frac{2}{t} + \frac{7 \ln t}{t}}$.

216. (a) Calculate the integral $\int 3x^2 \mathrm{d}x$. $\boxed{x^3 + C}$

(b) Calculate the integral $\int (x^3 + k) \mathrm{d}x$. (Your answer will include k.) $\boxed{\frac{1}{4}x^4 + kx + C}$

217. Solve $y'' = 3x^2$ by first finding $y' = \int (y'')\mathrm{d}x = \boxed{x^3 + C_1}$ and then finding $y = \int (y')\mathrm{d}x = \int (x^3 + C_1)\mathrm{d}x = \boxed{\frac{1}{4}x^4 + C_1 x + C_2}$

218. Classify each linear ODEs below as "homogeneous linear" or "non-homogeneous linear" or "not linear".

(a) $y'' - 9y' + 2y = 0$. homogeneous linear

(b) $y'' + 9y' + 2y = t^3$. non-homogeneous linear

(c) $y''' - y'' - y' - y = 0$. homogeneous linear

(d) $x'' + 2x = 9x'$. homogeneous linear

(e) $y'' = 5t$. non-hom. linear because $y'' + (0)y' + (0)y = (5t)$.

(f) $y'' = 5y$. hom. linear because $y'' + (0)y' + (-5)y = 0$.

(g) $x'' = 5x'$. hom. linear because $x'' + (-1)x' + (0)x = 0$.

(h) $y'' = 5t^2$. non-hom. linear because $y'' + (0)y' + (0)y = (5y^2)$.

(i) $y'' = 5x^2$. non-hom. linear because $y'' + (0)y' + (0)y = (5x^2)$.

(j) $y'' = 5y^2$. not linear

(k) $y'' = 5yt$. hom. linear because $y'' + (0)y' + (-5t)y = 0$.

A collection of functions $y_1(t), y_2(t), ..., y_k(t)$ are called a **fundamental set** of solutions for a homogeneous ODE if the general solution to the ODE is

$$y = C_1 \cdot y_1(t) + C_2 \cdot y_2(t) + \cdots + C_k y_k(t).$$

We can also[1] say that the functions are "fundamental solutions" to the ODE.

219. The functions t^{10} and $\frac{1}{t}$ form a fundamental set for

$$y'' - \frac{8}{t}y' - \frac{10}{t^2}y = 0.$$

Using this, give the general solution to that ODE. $\boxed{y = C_1 t^{10} + \frac{C_2}{t}}$

[1]Using linear algebra vocabulary that is not required for this class, the fundamental set is a "basis" for the solution space, the fundamental functions "span" the solutions space, and the general solution is the set of all "linear combinations" of the fundamental solutions.

☆220. Solve the ODE $y'' - \frac{8}{t}y' - \frac{10}{t^2}y = 0$.

Note: This task is starred. I would *not* ask you to solve this ODE on any quiz or exam because it is second-order but does *not* have constant coefficients. (But Task 219, with fundamental solution provided, is not starred.)

For a homogeneous linear ODE with constant coefficients

$$a_n y^{(n)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0,$$

the **characteristic polynomial** of this ODE is

$$a_n r^n + \cdots + a_2 r^2 + a_1 r + a_0.$$

221. For each of the following homogeneous linear ODEs with constant coefficients, write the characteristic polynomial and find its (real or complex) roots.

(a) $y'' + y' - 2y = 0$. $\boxed{r^2 + r - 2}$ has roots $\boxed{1 \text{ and } -2}$.

(b) $2y'' + y' - 21y = 0$. $\boxed{2r^2 + r - 21}$ has roots $\boxed{r = 3, r = \frac{-7}{2}}$

(c) $y''' = -5y'' - y' + 5y$. $\boxed{r^3 - 5r^2 + r - 5}$ has roots $\boxed{5,\ i\ \ -i}$.

☆(d) $y^{(4)} - 8y''' + 16y'' - 25y = 0$. $\boxed{x^4 - 8x^3 + 16x^2 - 25}$ has roots $\boxed{5,\ -1,\ 2+i,\ 2-i}$.

(e) $y^{(4)} - 4y''' + 5y'' = 0$. $\boxed{r^4 - 4r^3 + 5r^2}$ has roots $\boxed{0}$ (twice), $\boxed{2+i}$, and $\boxed{2-i}$.

(f) $x'' - 10x' + 9x = 0$. $\boxed{r^2 - 10r + 9}$ has roots $\boxed{1 \text{ and } 9}$

If the set of fundamental solutions for a homogeneous linear ODE with constant coefficients can be found using the roots of its characteristic polynomial:

- for each root $a \pm bi$ with multiplicity m, all of the functions $t^k e^{at}\sin(bt)$ and $t^k e^{at}\cos(bt)$ with $k = 0, 1, ..., m-1$ are fundamental solutions.

That one rule completely describes the fundamental set, but in practice the following rules are easier to use for second-order ODEs:

- if r_1 and r_2 distinct real roots, then use $e^{r_1 t}$ and $e^{r_2 t}$.
- if r is a repeated real root, then use e^{rt} and $t\,e^{rt}$.
- if $\lambda + \mu i$ is a complex root, then use $e^{\lambda t}\sin(\mu t)$ and $e^{\lambda t}\cos(\mu t)$.

222. Give a set of fundamental solutions for the ODE

$$y'' + 3y' - 18y = 0.$$

$\boxed{e^{3t} \text{ and } e^{-6t}}$ Answers such as "$\frac{1}{5}e^{-6t}$ and $-100e^{3t}$" are also correct.

223. Describe all possible solutions to the homogeneous ODE

$$y'' - 8y' + 25y = 0.$$

Polynomial $\lambda^2 - 8\lambda + 25$ has roots $4 \pm 3i$, so $\boxed{y = C_1 e^{4t}\cos(3t) + C_2 e^{4t}\sin(3t)}$.

224. Find the general solution to the following homogeneous linear ODEs:

(a) $y'' + y' - 2y = 0$. Char. polyn. $f(r) = r^2 + r - 2$ has roots 1 and -2, so the general solution is $\boxed{C_1e^t + C_2e^{-2t}}$.

(b) $y'' + 2y' + y = 0$. Char. polyn. $f(r) = r^2 + 2r + 1$ has repeated root -1, so the general solution is $\boxed{C_1e^{-t} + C_2te^{-t}}$.

(c) $y''' + 3y'' - 4y' = 0$. Char. polyn. $f(r) = r^3 + 3r^2 - 4r$ has roots 0, 1, and -4, so the general solution is $C_1e^{0t} + C_2e^{1t} + C_3e^{-4t} = \boxed{C_1 + C_2e^t + C_3e^{-4t}}$.

(d) $y''' + y'' + y' + y = 0$. Char. polyn. $f(r) = r^3 + r^2 + r + 1$ has roots -1 and $0+i$ and $0-i$, so the general solution is $C_1e^{-t} + C_2e^{0t}\sin(t) + C_3e^{0t}\cos(t) = \boxed{C_1e^{-t} + C_2\sin(t) + C_3\cos(t)}$.

(e) $y^{(4)} - 5y'' + 4y = 0$. $f(r) = r^4 - 5r^2 + 4$ has roots 1, -1, 2, -2, so the general solution is $\boxed{C_1e^t + C_2e^{-t} + C_3e^{2t} + C_4e^{-2t}}$.

(f) $y^{(4)} - 8y''' + 16y'' - 25y = 0$. $f(r) = r^4 - 8r^3 + 16r^2 - 25$ has roots 5, -1, $2+i$, $2-i$, so $y = \boxed{C_1e^{5t} + C_2e^{-t} + C_3e^{2t}\sin(t) + C_4e^{2t}\cos(t)}$.

(g) $y^{(4)} + 8y'' + 16y = 0$. $f(r) = r^4 + 8r^2 + 16$ has repeated roots $2i$ (twice) and $-2i$ (twice), so $y = \boxed{C_1\cos(2t) + C_2\,t\cos(2t) + C_3\sin(2t) + C_4\,t\sin(2t)}$.

(h) $y^{(5)} + y''' = 0$. $\boxed{y = C_1 + C_2t + C_3t^2 + C_4\sin(t) + C_5\cos(t)}$

225. Solve the IVP

$$y'' + 2y' + y = 0,\ y(0) = 7,\ y'(0) = 5.$$

From 224(b), $y = C_1e^{-t} + C_2te^{-t}$. Thus $y(0) = C_1 = 7$ and since

$$y'(t) = \left(C_1e^{-t} + C_2te^{-t}\right)' = -7e^{-t} - C_2te^{-t} + C_2e^{-t}$$

we have $y'(0) = -7 - 0 + C_2 = 5$, so $C_2 = 12$. Thus $\boxed{y = 7e^{-t} + 12te^{-t}}$.

226. Using the fact that $r^3 - r^2 + r - 1 = (r-1)(r^2+1)$,

(a) Solve the ODE $y''' - y'' + y' - y = 0$. $\boxed{y = C_1e^t + C_2\cos(t) + C_3\sin(t)}$

(b) Solve the IVP $y''' - y'' + y' - y = 0$, $y(0) = 5$, $y'(0) = -3$, $y''(0) = 1$. $\boxed{y = 3e^t + 2\cos(t) - 6\sin(t)}$

227. Give the homogeneous linear ODE with constant coefficients for which

$$y = C_1e^{-t} + C_2e^t + C_3te^t + C_4t^2e^t + C_5e^{4t}\sin(3t) + C_6e^{4t}\cos(3t)$$

is the general solution.

$(r+1)(r-1)^3(r-(4+3i))(r-(4-3i)) = r^6 - 10r^5 + 41r^4 - 48r^3 - 17r^2 + 58r - 25$, so the ODE is $\boxed{y^{(6)} - 10y^{(5)} + 41y^{(4)} - 48y''' - 17y'' + 58y' - 25y = 0}$.

228. (a) Find the unique value of A for which $y = Ae^{-2t}$ is a solution to

$$y'' - 8y' + 25y = 50e^{-2t}.$$

$$\begin{aligned}(Ae^{-2t})'' - 8(Ae^{-2t})' + 25(Ae^{-2t}) &= 50e^{-2t}\\ (4Ae^{-2t}) - 8(-2Ae^{-2t}) + 25(Ae^{-2t}) &= 50e^{-2t}\\ 4Ae^{-2t} + 16Ae^{-2t} + 25Ae^{-2t} &= 50e^{-2t}\\ 4A + 16A + 25A &= 50\\ A = \frac{50}{45} &= \boxed{\frac{10}{9}}\end{aligned}$$

(b) Give one particular solution to the ODE. $\boxed{y = \frac{10}{9}e^{-2t}}$

Given one particular solution $y = y_{\text{NH}}(t)$ to a non-homogeneous linear ODE, the general solution will be

$$y = y_{\text{NH}} + y_{\text{Hom}},$$

where y_{Hom} solves the corresponding homogeneous ODE.

The format of y_{NH} depends on the non-homogeneous term $f(t)$. If $\lambda + \omega i$ is not a root of the characteristic polynomial, then

$$\begin{aligned} f &= ae^{\lambda t} &\Rightarrow\quad & y_{\text{NH}} = Ae^{\lambda t}\\ f &= a\sin(\omega t) &\Rightarrow\quad & y_{\text{NH}} = A\sin(\omega t) + B\cos(\omega t)\\ f &= a\cos(\omega t) &\Rightarrow\quad & y_{\text{NH}} = A\sin(\omega t) + B\cos(\omega t)\\ f &= at^k &\Rightarrow\quad & y_{\text{NH}} = At^k + \cdots + Yt + Z \end{aligned}$$

where $A, B, \ldots$ are unknown numbers. If f is a sum or product of terms on the left, then y_{NH} should be a sum or product of formulas on the right.

If $\lambda + \omega i$ is a root of the characteristic polynomial (in this case we say the ODE has **resonance**) with multiplicity m, then multiply the suggested y_{NH} above by t^m.

229. Using Task 228(b), describe all possible solutions to the ODE

$$y'' - 8y' + 25y = 50e^{-2t}.$$

This is exactly the answer to Task 223 plus the answer to Task 228(b).

That is, $\boxed{y = \frac{10}{9}e^{-2t} + C_1e^{4t}\cos(3t) + C_2e^{4t}\sin(3t)}$.

230. Write the form (use capital letters $A, B, \ldots$ for any unknown coefficients, and assume there is no resonance) of the non-homogeneous part of the solution to the constant-coefficient linear ODE $a_k y^{(k)} + \cdots + a_0 y = f(t)$ if...

(a) $f(t) = e^{4t}$. $\boxed{Ae^{4t}}$

(b) $f(t) = \cos(2t)$. $\boxed{A\cos(2t) + B\sin(2t)}$

(c) $f(t) = \cos(2t) + \sin(2t)$. $\boxed{A\cos(2t) + B\sin(2t)}$ (same as (b))

(d) $f(t) = \cos(2t) + \sin(3t)$. $\boxed{A\cos(2t) + B\sin(2t) + C\cos(3t) + D\sin(3t)}$

(e) $f(t) = e^{9t} + 7$. $\boxed{Ae^{9t} + (Bt + C)}$

(f) $f(t) = t^4 + \sin(t)$. $\boxed{(At^4 + Bt^3 + Ct^2 + Dt + E) + F\cos(t) + G\sin(t)}$

(g) $f(t) = t^3\sin(6t)$. $\boxed{(A + Bt + Ct^2 + Dt^3)\sin(6t) + (E + Ft + Gt^2 + Ht^3)\cos(6t)}$
or $A\sin(6t) + Bt\sin(6t) + Ct^2\sin(6t + Dt^3\sin(6t) + E\cos(6t) + Ft\cos(6t) + Gt^2\cos(6t) + Ht^3\cos(6t)$

231. Give the form of y_{NH} for

(a) $y'' - 2y' - 24y = e^{4t}$. $\boxed{A\,e^{4t}}$ (See Task 230(a).)

(b) $y'' + 2y' - 24y = e^{4t}$. $\boxed{A\,t\,e^{4t}}$ where the extra t is because of resonance. (This is **not** the same as the answer to part (a) because of resonance.)

(c) $y'' - 8y' + 16y = e^{4t}$. $\boxed{A\,t^2\,e^{4t}}$ (Double-resonance!)

232. Solve the non-homogeneous linear ODE

$$y'' - 2y' - 2y = 26e^{5t}.$$

Corresponding homogeneous has solutions $e^{(1+\sqrt{3})t}$ and $e^{(1-\sqrt{3})t}$ because $f(r) = r^2 - 2r - 2$ has roots $1 \pm \sqrt{3}$. Because $g(t) = 26e^{5t}$, the non-homogeneous part has the form

$$y_{\mathrm{NH}} = Ae^{5t}.$$

Using the ODE, we have

$$\begin{aligned}
(Ae^{5t})'' - 2(Ae^{5t})' - 2(Ae^{5t}) &= 26e^{5t} \\
(25Ae^{5t}) - 2(5Ae^{5t}) - 2(Ae^{5t}) &= 26e^{5t} \\
25Ae^{5t} - 10Ae^{5t} - 2Ae^{5t} &= 26e^{5t} \\
13Ae^{5t} &= 26e^{5t} \\
13A &= 26 \\
A &= 2
\end{aligned}$$

Therefore $y_{\mathrm{NH}} = 2e^{5t}$ and $\boxed{y = C_1e^{(1+\sqrt{3})t} + C_2e^{(1-\sqrt{3})t} + 2e^t}$.

233. Solve the IVP

$$\tfrac{1}{2}y'' - 5y' + 8y = 0, \quad y(0) = 5,\ y'(0) = 22$$

(a) by first solving the ODE and then finding C_1, C_2. General solution: $y = C_1e^{2t} + C_2e^{8t}$. From $y(0) = 5$ we get $5 = C_1 + C_2$. To use the second IC we first need to take a derivative:

$$y' = \left(C_1e^{2t} + C_2e^{8t}\right)' = 2C_1e^{2t} + 8C_2e^{8t}.$$

So $y'(0) = 22$ means $2C_1 + 8C_2 = 22$. The system

$$\begin{cases} C_1 + C_2 \quad = 5 \\ 2C_1 + 8C_2 = 22 \end{cases}$$

has solution $(C_1, C_2) = (3, 2)$, so the particular solution is $\boxed{y(t) = 3e^{2t} + 2e^{8t}}$.

(b) by using Laplace transforms. For this you will need to use the fact that

$$\mathscr{L}\big[y''\big] = s^2 \cdot Y - s \cdot y(0) - y'(0).$$

This is a consequence of the rule $\mathscr{L}\big[f'\big] = s \cdot F - f(0)$, which we have seen before, with $f = y$. Laplace transforms turn this into $Y(s) = \dfrac{5s-28}{s^2-10s+16}$, which by **Tasks 202 or 205(e)** gives $\boxed{y(t) = 3e^{2t} + 2e^{8t}}$.

234. Solve the IVP

$$x'' - x' = (1+t)\sin t,\ \ x(0) = 0,\ \ x'(0) = 1.$$

Homogeneous solutions $e^{0t} = 1$ and e^t.
Particular solution $x_{\mathrm{NH}} = \frac{-3}{2}\sin(t) - \frac{1}{2}t\sin(t) + \frac{1}{2}t\cos(t)$. General solution

$$x = C_1 + C_2e^t - \tfrac{3}{2}\sin(t) - \tfrac{1}{2}t\sin(t) + \tfrac{1}{2}t\cos(t).$$

See Task 9: initial conditions lead to $C_1 = -2$ and $C_2 = 2$, so

$$\boxed{x = -2 + 2e^t + \tfrac{-3}{2}\sin(t) - \tfrac{1}{2}t\sin(t) + \tfrac{1}{2}t\cos(t)}.$$

Alternatively, Laplace gives $s^2X(s) - sX(s) - 1 = \dfrac{2s}{(s^2+1)^2} + \dfrac{1}{s^2+1}$ and then

$$X = \frac{s^4+3s^2+2s+2}{(s-1)s\,(s^2+1)^2} = \frac{-s-1}{(s^2+1)^2} - \frac{1}{s^2+1} + \frac{2}{s-1} - \frac{2}{s},$$

so $\boxed{x = -2 + 2e^t + \tfrac{-3}{2}\sin(t) - \tfrac{1}{2}t\sin(t) + \tfrac{1}{2}t\cos(t)}$.

235. Solve the following higher-order ODEs (they all have constant coefficients):

(a) $y'' - 4y' - 60y = 0$ $\boxed{y = C_1e^{10t} + C_2e^{6t}}$

(b) $y'' - 10y' + 23y = 0$ $\boxed{y = C_1e^{(5-\sqrt{2})t} + C_2e^{(5+\sqrt{2})t}}$

(c) $y'' + 8y' + 17y = 0$ $\boxed{y = C_1e^{-4t}\sin(t) + C_2e^{-4t}\cos(t)}$

(d) $x'' + 7x' + 10x = 0$ $\boxed{x = C_1e^{5t} + C_2e^{-2t}}$

(e) $y'' - y' - 12y = 0$ $\boxed{y = C_1e^{-3t} + C_2e^{4t}}$

(f) $y'' - y' - 12y = 13e^{10t}$
y_{Hom} is from part (e), and $y_{\mathrm{NH}} = Ae^{10t}$ for some A.
Using $(Ae^{10t})'' - (Ae^{10t})' - 12(Ae^{10t}) = 13e^{10t}$ we get $A = \frac{1}{6}$,
so $\boxed{y = C_1e^{-3t} + C_2e^{4t} + \tfrac{1}{6}e^{10t}}$

(g) $x'' - 4x' + 13x = 0$ $\boxed{x = C_1e^{2t}\sin(3t) + C_2e^{2t}\cos(3t)}$

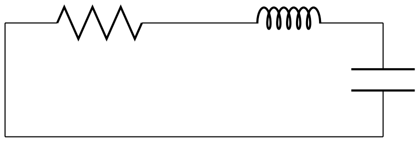
(h) $x'' + 3x' + 2x = 4t^2 - 11$ $\boxed{x = C_1e^{-2t} + C_2e^{-t} + 2t^2 - 6t + \dfrac{3}{2}}$

(i) $y'' - 2y' + 82y = 0$ $\boxed{y = C_1e^{t}\sin(9t) + C_2e^{t}\cos(9t)}$

(j) $y'' - y' = 8\sin(t)$ $\boxed{y = 4\cos(t) - 4\sin(t) + C_1 e^t + C_2}$

(k) $y''' - 6y'' + 5y' = 0$ $\boxed{y = C_1 + C_2 e^t + C_3 e^{5t}}$

236. *RLC circuit:* The current $i(t)$ in the circuit



satisfies the second-order differential equation

$$\frac{\mathrm{d}^2 i}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{LC} i = 0.$$

Using $R = 6\ \Omega$, $L = 2$ H, and $C = 0.04$ F, solve the IVP

$$i'' + 3i' + \frac{25}{2} i = 0, \qquad i(0) = \frac{1}{10} \text{ ampere}, \quad i'(0) = 0 \ \frac{\text{ampere}}{\text{second}}.$$

$$\boxed{\begin{aligned} i &= \frac{1}{10} e^{-3t/2} \cos(\tfrac{\sqrt{41}}{2} t) + \frac{3}{10\sqrt{41}} e^{-3t/2} \sin(\tfrac{\sqrt{41}}{2} t) \\ &= 0.1\, e^{-1.5t} \cos(3.2t) + 0.047\, e^{-1.5t} \sin(3.2t) \end{aligned}}$$

237. Solve the IVP $y'' - 9y = -32\, t\, e^t$, $\quad y(0) = 5,\ y'(0) = \frac{1}{2}$.

Method 1: Characteristic polynomials. The homogeneous ODE $y'' - 9y = 0$ has char. polyn. $r^2 - 9$, which has roots ± 3, so $y_{\text{Hom}} = C_1 e^{3t} + C_2 e^{-3t}$. The non-homogeneous term is a product of a polynomial and an exponential, so y_{NH} must also be the product of a polynomial (same degree as "t", so degree 1, so some $At + B$) and an exponential (same exponent as e^t).

$$y_{\text{NH}} = (At + B)e^t \qquad \text{for some } A, B.$$

We don't need Ce^t because $(At+B)e^t = Ate^t + Be^t$ already includes multiplying e^t by a number.

$$\begin{aligned} y''_{\text{NH}} - 9y_{\text{NH}} &= -32\, t\, e^t \\ (Ate^t + Be^t)'' - 9(Ate^t + Be^t) &= -32\, t\, e^t \\ (2Ae^t + Ate^t + Be^t) + (-9Ate^t - 9Be^t) &= -32\, t\, e^t \\ 2AE^t - 8AtE^t - 8BE^t &= -32\, t\, e^t \\ (-8A)t + (2A - 8B) &= -32\, t \end{aligned}$$

Therefore $\begin{cases} -8A = -32 \\ 2A - 8B = 0 \end{cases}$ and so $A = 4$, $B = 1$, and $y_{\text{NH}} = (4t + 1)e^t$. The general solution is $\boxed{y = C_1 e^{3t} + C_2 e^{-3t} + 4te^t + e^t}$.

Before we can use the initial condition $y'(0) = \frac{1}{2}$, we will need

$$\begin{aligned} y' &= \left(C_1 e^{3t} + C_2 e^{-3t} + 4te^t + e^t\right)' \\ &= 3C_1 e^{3t} - 3C_2 e^{-3t} + (4te^t + 4e^t) + e^t \\ &= 3C_1 e^{3t} - 3C_2 e^{-3t} + 4te^t + 5e^t \end{aligned}$$

The initial conditions give us the system

$$\begin{array}{r} y(0) = C_1 + C_2 + 1 = 5 \\ y'(0) = 3C_1 - 3c_2 + 5 = \frac{1}{2} \end{array} \quad \rightarrow \quad \begin{cases} C_1 + \ \ C_2 = 4 \\ 3C_1 - 3C_2 = \frac{-9}{2} \end{cases}$$

This can be solved many ways. Using Cramer's Rule,

$$C_1 = \frac{\det \begin{bmatrix} 4 & 1 \\ \frac{-9}{2} & -3 \end{bmatrix}}{\det \begin{bmatrix} 1 & 1 \\ 3 & -3 \end{bmatrix}} = \frac{\frac{-15}{2}}{-6} = \frac{5}{4} \quad \text{and} \quad C_2 = \frac{\det \begin{bmatrix} 1 & 4 \\ 3 & \frac{-9}{2} \end{bmatrix}}{\det \begin{bmatrix} 1 & 1 \\ 3 & -3 \end{bmatrix}} = \frac{\frac{-33}{2}}{-6} = \frac{11}{4}.$$

The final answer is therefore $\boxed{y = \frac{5}{4}e^{3t} + \frac{11}{4}e^{-3t} + 4te^t + e^t}$.

Method 2: Laplace transforms. We need $\mathscr{L}\big[t\,e^t\big]$. Using $\mathscr{L}\big[e^t f\big] = F(s-1)$ with $f(t) = t$, we get $F(s) = \mathscr{L}\big[t\big] = \frac{1}{s^2}$ and so $F(s-1) = \frac{1}{(s-1)^2}$. Alternatively, using $\mathscr{L}\big[t\,f\big] = -F'$ with $f = e^t$, we get $F(s) = \mathscr{L}\big[e^t\big] = \frac{1}{s-1}$ and then $-F' = -(\frac{-1}{(s-1)^2}) = \frac{1}{(s-1)^2}$. Taking the Laplace transform of the entire IVP and then solving for Y, we find

$$\begin{aligned} \mathscr{L}\big[y''\big] - 9\mathscr{L}\big[y\big] &= -32\mathscr{L}\big[t\,e^t\big] \\ s^2Y - sy(0) - y'(0) - 9Y &= -32 \cdot \frac{1}{(s-1)^2} \\ s^2Y - 5s - \tfrac{1}{2} - 9Y &= \frac{-32}{(s-1)^2} \\ (s^2-9)Y &= \frac{-32}{(s-1)^2} + 5s + \frac{1}{2} \\ (s^2-9)Y &= \frac{10s^3 - 19s^2 + 8s - 63}{2(s-1)^2} \\ Y &= \frac{10s^3 - 19s^2 + 8s - 63}{2(s-3)(s+3)(s-1)^2}. \end{aligned}$$

The partial fraction decomposition for Y is

$$Y = \frac{A}{s-3} + \frac{B}{s+3} + \frac{C}{(s-1)} + \frac{D}{(s-1)^2}$$

for some A, B, C, D. We could combine the last two terms into $\frac{Cs+E}{(s-1)^2}$, but (1) this would not match the official definition of partial fractions and (2) we would have to split it apart later anyway when doing inverse Laplace transforms.

Multiplying everything by $2(s-3)(s+3)(s-1)^2$ gives

$$\begin{aligned} 10s^3 - 19s^2 + 8s - 63 \ &= \ 2A(s+3)(s-1)^2 + 2B(s-3)(s-1)^2 \\ &\qquad + 2C(s+3)(s-3)(s-1) + 2D(s+3)(s-3) \end{aligned}$$

One method to find A, B, C, D would be to expand the right hand side completely and then group terms to get

$$10s^3 - 19s^2 + 8s - 63 = (2A{+}2B{+}2C)s^3 + (2A{-}10B{-}2C{+}2D)s^2 + \cdots$$

and compare like terms ($10 = 2A{+}2B{+}2C$, etc.).

Alternatively, we can plug in specific numbers to get simple equations.

$$s = 3 \quad \to \quad 60 = 48A + 0 + 0 + 0$$
$$s = -3 \quad \to \quad -528 = 0 + 0 - 192B + 0$$
$$s = 1 \quad \to \quad -64 = 0 + 0 + 0 - 16D$$

and so $A = \frac{60}{48} = \frac{5}{4}$ and $B = \frac{-528}{-192} = \frac{11}{4}$ and $D = 4$. To get C we need to plug in any number besides ± 3 or $+1$. For example,

$$s = 0 \quad \to \quad -63 = 6A - 6B + 18C - 18D$$
$$-63 = 18c - 81$$

so $C = 1$ and

$$Y = \frac{\frac{5}{4}}{s-3} + \frac{\frac{11}{4}}{s+3} + \frac{1}{(s-1)} + \frac{4}{(s-1)^2}$$
$$\boxed{y = \frac{5}{4}e^{3t} + \frac{11}{4}e^{-3t} + e^t + 4te^t}$$

238. Complete the following table:

Linear ODE	**Constant coefficients?**	**Homogeneous?**
$ty'' + \sin(t)y = 0$	no	yes
$y'' - 5y' - y = 0$	yes	yes
$y'' - 5y' = y$	yes	yes
$x'' + tx' - 7x = 0$	no	yes
$x'' = x + t$	yes	no
$x' = \cos(t)$	yes	no
$x' = \cos(t)x$	no	yes

239. Solve the ODE

$$y' + 17y = 0$$

for $y(t)$ using...

(a) separation of variables (this is a separable ODE).

$$\frac{dy}{dt} = 17y \quad \Rightarrow \quad \int \frac{dy}{y} = \int 17 dt \quad \Rightarrow \quad \ln|y| = 17t + C \quad \Rightarrow \quad \boxed{y = Ce^{17t}}$$

(b) characteristic polynomials (this is a homogeneous linear ODE with constant coefficients).

The characteristic equation is

$$r - 17 = 0,$$

so $r = 17$ is the only root and e^{17t} is the only fundamental solution. The general solution is therefore $\boxed{y = Ce^{17t}}$.

240. Solve the IVP

$$x' + 3x = 8, \qquad x(0) = 9$$

in several ways:

(a) Separation of variables (this is separable).

$$\frac{dx}{dt} + 3x = 8 \quad \to \quad \int \frac{dx}{-3x+8} = dt$$

$$\to \quad \tfrac{-1}{3}\ln(-3x+8) = t + C \quad \to \quad \boxed{x = Ce^{-3t} + \tfrac{8}{3}}.$$

From $x(0) = 9$ we get $C = \frac{19}{3}$ and so $\boxed{x = \frac{19}{3}e^{-3t} + \frac{8}{3}}$.

(b) Variation of parameters (this is first-order linear).

$x' + 3x = 0$ would have solution Ce^{-3t}, so assume $x = g \cdot e^{-3t}$. Then

$$(ge^{-3t})' + 3(ge^{-3t}) = 8 \quad \to \quad g' = \frac{8}{e^{-3t}} = 8e^{3t} \quad \to \quad g = \tfrac{8}{3}e^{3t} + C$$

and $x = g \cdot e^{-3t} = (\frac{8}{3}e^{3t} + C) \cdot e^{-3t} = \boxed{\frac{8}{3} + Ce^{-3t}}$.

From $x(0) = 9$ we get $C = \frac{19}{3}$ and so $\boxed{x = \frac{19}{3}e^{-3t} + \frac{8}{3}}$.

(c) Integrating factor (this is first-order linear).

Version 1: Multiply the ODE by a completely unknown function $M = M(t)$.

$$Mx' + 3Mx = 8M$$

The left-hand side is similar to $(Mx)' = Mx' + M'x$, and it is exactly this if $M' = 3M$. So we can use $M = e^{3t}$ to make $(Mx)'$ appear.

$$e^{3t}x' + 3e^{3t}x = 8e^{3t} \qquad (*)$$

$$(e^{3t}x)' = 8e^{3t}$$

$$e^{3t}x = \tfrac{8}{3}e^{3t} + C$$

$$x = (\tfrac{8}{3}e^{3t} + C)e^{-3t}$$

$$\boxed{x = \tfrac{8}{3} + Ce^{-3t}}$$

From $x(0) = 9$ we get $C = \frac{19}{3}$ and so $\boxed{x = \frac{19}{3}e^{-3t} + \frac{8}{3}}$.

Version 2: Since $a(t) = 3$ in the original ODE, we will have

$$M = e^{A(t)} = e^{(\int 3dt)} = e^{3t}.$$

Multiplying by e^{3t} gives exactly the $(*)$ equation

$$e^{3t}x' + 3e^{3t}x = 8e^{3t},$$

from above, and the rest of the process is exactly like Version 1.

(d) The "big formula" for first-order linear ODEs.

Using $a(t) = 3$ and $f(t) = 8$, we have $A(t) = 3t$ and

$$x = \left(\int e^{A} f dt\right) e^{-A} = \left(\int 8e^{3t} dt\right) e^{-3t} = (\tfrac{8}{3}e^{3t} + C)e^{-3t} = \boxed{\tfrac{8}{3} + Ce^{-3t}}.$$

From $x(0) = 9$ we get $C = \frac{19}{3}$ and so $\boxed{x = \frac{8}{3} + \frac{19}{3}e^{-3t}}$.

(e) Laplace transformations.

Using either

$$\mathscr{L}[8] = 8\mathscr{L}[1] = 8\mathscr{L}[e^{0t}] = 8\frac{1}{s-0} = \frac{8}{s}$$

or

$$\mathscr{L}[8] = 8\mathscr{L}[1] = 8\mathscr{L}[t^0] = 8\frac{0!}{s^{0+1}} = \frac{8}{s},$$

we get

$$\begin{aligned}
\mathscr{L}[x'] + 3\mathscr{L}[x] &= \mathscr{L}[8] \\
(sX - 9) + 3X &= \frac{8}{s} \\
X &= \frac{9s+8}{s(s+3)} \quad \text{just from solving for } X \\
X &= \frac{8/3}{s} + \frac{19/3}{s+3} \quad \text{as a sum of partial fractions} \\
x &= \frac{8}{3}\mathscr{L}^{-1}\left[\frac{1}{s}\right] + \frac{19}{3}\mathscr{L}^{-1}\left[\frac{1}{s+3}\right] \\
&\boxed{x = \tfrac{8}{3} + \tfrac{19}{3}e^{-3t}}
\end{aligned}$$

(f) Characteristic polynomials (this is a non-homogeneous linear IVP with constant coefficients, so you will also need x_{NH} for the polynomial $g(t) = 8$).

The characteristic polynomial gives $r + 3 = 0$, so $r = -3$ and the homogeneous part of the solution is $x = Ce^{-3t}$.

From $g = 8$ (which we can think of as $8t^0$ or as $8e^{0t}$) we get that x_{NH} is some constant function. Writing $x_{\mathrm{NH}} = A$, we have

$$\begin{aligned}
(A)' + 3(A) &= 8 \\
0 + 3A &= 8 \\
A &= \tfrac{8}{3}
\end{aligned}$$

So $x_{\mathrm{NH}} = \frac{8}{3}$ and $x = x_{\mathrm{Hom}} + x_{\mathrm{NH}} = \boxed{Ce^{-3t} + \frac{8}{3}}$.

From $x(0) = 9$ we get $C = \frac{19}{3}$ and so $\boxed{x = \frac{19}{3}e^{-3t} + \frac{8}{3}}$.