Analysis 2, Summer 2024

List 9

Higher-order linear ODEs

☆214. Solve the ODE $t^2y'' + 3ty' + y = 0$.

Note: This task is starred. I would *not* ask you to solve this ODE on any quiz or exam because it is second-order but does *not* have constant coefficients.

215. The second-order ODE

$$t^2y'' + 3ty' + y = 0$$

has general solution

$$y = \frac{C_1}{t} + \frac{C_2 \ln t}{t}.$$

(a) Use this general solution to find a formula for y'.

(b) Using the formulas for y and y', give the solution to the IVP

$$t^2y'' + 3ty' + y = 0, \quad y(1) = 2, \quad y'(1) = 5.$$

Note: This is the same ODE has Task 214, but this task is not starred.

216. (a) Calculate the integral $\int 3x^2 \mathrm{d}x$.

(b) Calculate the integral $\int (x^3 + k)\mathrm{d}x$. (Your answer will include k.)

217. Solve $y'' = 3x^2$ by first finding $y' = \int (y'')\mathrm{d}x$ and then finding $y = \int (y')\mathrm{d}x$.

218. Classify each linear ODEs below as "homogeneous linear" or "non-homogeneous linear" or "not linear".

(a) $y'' - 9y' + 2y = 0$.

(b) $y'' + 9y' + 2y = t^3$.

(c) $y''' - y'' - y' - y = 0$.

(d) $x'' + 2x = 9x'$.

(e) $y'' = 5t$.

(f) $y'' = 5y$.

(g) $x'' = 5x'$.

(h) $y'' = 5t^2$.

(i) $y'' = 5x^2$.

(j) $y'' = 5y^2$.

(k) $y'' = 5yt$.

A collection of functions $y_1(t), y_2(t), ..., y_k(t)$ are called a **fundamental set** of solutions for a homogeneous ODE if the general solution to the ODE is

$$y = C_1 \cdot y_1(t) + C_2 \cdot y_2(t) + \cdots + C_k y_k(t).$$

We can also[1] say that the functions are "fundamental solutions" to the ODE.

219. The functions t^{10} and $\frac{1}{t}$ form a fundamental set for

$$y'' - \frac{8}{t}y' - \frac{10}{t^2}y = 0.$$

Using this, give the general solution to that ODE.

[1] Using linear algebra vocabulary that is not required for this class, the fundamental set is a "basis" for the solution space, the fundamental functions "span" the solutions space, and the general solution is the set of all "linear combinations" of the fundamental solutions.

☆220. Solve the ODE $y'' - \frac{8}{t}y' - \frac{10}{t^2}y = 0$.

Note: This task is starred. I would *not* ask you to solve this ODE on any quiz or exam because it is second-order but does *not* have constant coefficients. (But Task 219, with fundamental solution provided, is not starred.)

> For a homogeneous linear ODE with constant coefficients
> $$a_n y^{(n)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0,$$
> the **characteristic polynomial** of this ODE is
> $$a_n r^n + \cdots + a_2 r^2 + a_1 r + a_0.$$

221. For each of the following homogeneous linear ODEs with constant coefficients, write the characteristic polynomial and find its (real or complex) roots.

(a) $y'' + y' - 2y = 0$.

(b) $2y'' + y' - 21y = 0$.

(c) $y''' = -5y'' - y' + 5y$.

☆(d) $y^{(4)} - 8y''' + 16y'' - 25y = 0$.

(e) $y^{(4)} - 4y''' + 5y'' = 0$.

(f) $x'' - 10x' + 9x = 0$.

> If the set of fundamental solutions for a homogeneous linear ODE with constant coefficients can be found using the roots of its characteristic polynomial:
>
> - for each root $a \pm bi$ with multiplicity m, all of the functions $t^k e^{at}\sin(bt)$ and $t^k e^{at}\cos(bt)$ with $k = 0, 1, ..., m-1$ are fundamental solutions.
>
> That one rule completely describes the fundamental set, but in practice the following rules are easier to use for second-order ODEs:
>
> - if r_1 and r_2 distinct real roots, then use $e^{r_1 t}$ and $e^{r_2 t}$.
> - if r is a repeated real root, then use e^{rt} and $t\,e^{rt}$.
> - if $\lambda + \mu i$ is a complex root, then use $e^{\lambda t}\sin(\mu t)$ and $e^{\lambda t}\cos(\mu t)$.

222. Give a set of fundamental solutions for the ODE
$$y'' + 3y' - 18y = 0.$$

223. Describe all possible solutions to the homogeneous ODE
$$y'' - 8y' + 25y = 0.$$

224. Find the general solution to the following homogeneous linear ODEs:

(a) $y'' + y' - 2y = 0$.

(b) $y'' + 2y' + y = 0$.

(c) $y''' + 3y'' - 4y' = 0$.

(d) $y''' + y'' + y' + y = 0$.

(e) $y^{(4)} - 5y'' + 4y = 0$.

(f) $y^{(4)} - 8y''' + 16y'' - 25y = 0$.

(g) $y^{(4)} + 8y'' + 16y = 0$.

(h) $y^{(5)} + y''' = 0$.

225. Solve the IVP
$$y'' + 2y' + y = 0,\ y(0) = 7,\ y'(0) = 5.$$

226. Using the fact that $r^3 - r^2 + r - 1 = (r-1)(r^2+1)$,

(a) Solve the ODE $y''' - y'' + y' - y = 0$.

(b) Solve the IVP $y''' - y'' + y' - y = 0,\ y(0) = 5,\ y'(0) = -3,\ y''(0) = 1$.

227. Give the homogeneous linear ODE with constant coefficients for which

$$y = C_1 e^{-t} + C_2 e^t + C_3 t e^t + C_4 t^2 e^t + C_5 e^{4t}\sin(3t) + C_6 e^{4t}\cos(3t)$$

is the general solution.

228. (a) Find the unique value of A for which $y = Ae^{-2t}$ is a solution to

$$y'' - 8y' + 25y = 50e^{-2t}.$$

(b) Give one particular solution to the ODE.

Given one particular solution $y = y_{\mathrm{NH}}(t)$ to a non-homogeneous linear ODE, the general solution will be

$$y = y_{\mathrm{NH}} + y_{\mathrm{Hom}},$$

where y_{Hom} solves the corresponding homogeneous ODE.

The format of y_{NH} depends on the non-homogeneous term $f(t)$. If $\lambda + \omega i$ is not a root of the characteristic polynomial, then

$$\begin{array}{lcl} f = ae^{\lambda t} & \Rightarrow & y_{\mathrm{NH}} = Ae^{\lambda t} \\ f = a\sin(\omega t) & \Rightarrow & y_{\mathrm{NH}} = A\sin(\omega t) + B\cos(\omega t) \\ f = a\cos(\omega t) & \Rightarrow & y_{\mathrm{NH}} = A\sin(\omega t) + B\cos(\omega t) \\ f = at^k & \Rightarrow & y_{\mathrm{NH}} = At^k + \cdots + Yt + Z \end{array}$$

where $A, B, \ldots$ are unknown numbers. If f is a sum or product of terms on the left, then y_{NH} should be a sum or product of formulas on the right.

If $\lambda + \omega i$ is a root of the characteristic polynomial (in this case we say the ODE has **resonance**) with multiplicity m, then multiply the suggested y_{NH} above by t^m.

229. Using Task 228(b), describe all possible solutions to the ODE

$$y'' - 8y' + 25y = 50e^{-2t}.$$

230. Write the form (use capital letters $A, B, \ldots$ for any unknown coefficients, and assume there is no resonance) of the non-homogeneous part of the solution to the constant-coefficient linear ODE $a_k y^{(k)} + \cdots + a_0 y = f(t)$ if...

(a) $f(t) = e^{4t}$.

(b) $f(t) = \cos(2t)$.

(c) $f(t) = \cos(2t) + \sin(2t)$.

(d) $f(t) = \cos(2t) + \sin(3t)$.

(e) $f(t) = e^{9t} + 7$.

(f) $f(t) = t^4 + \sin(t)$.

(g) $f(t) = t^3\sin(6t)$.

231. Give the form of y_{NH} for

(a) $y'' - 2y' - 24y = e^{4t}$.

(b) $y'' + 2y' - 24y = e^{4t}$. (This is **not** the same as the answer to part (a).)

(c) $y'' - 8y' + 16y = e^{4t}$.

232. Solve the non-homogeneous linear ODE

$$y'' - 2y' - 2y = 26e^{5t}.$$

233. Solve the IVP

$$\tfrac{1}{2}y'' - 5y' + 8y = 0, \quad y(0) = 5,\ y'(0) = 22$$

(a) by first solving the ODE and then finding C_1, C_2.

(b) by using Laplace transforms. For this you will need to use the fact that

$$\mathscr{L}\big[y''\big] = s^2 \cdot Y - s \cdot y(0) - y'(0).$$

This is a consequence of the rule $\mathscr{L}\big[f'\big] = s \cdot F - f(0)$, which we have seen before, with $f = y'$.

234. Solve the IVP

$$x'' - x' = (1+t)\sin t,\ x(0) = 0,\ x'(0) = 1.$$

235. Solve the following higher-order ODEs (they all have constant coefficients):

(a) $y'' - 4y' - 60y = 0$

(b) $y'' - 10y' + 23y = 0$

(c) $y'' + 8y' + 17y = 0$

(d) $x'' + 7x' + 10x = 0$

(e) $y'' - y' - 12y = 0$

(f) $y'' - y' - 12y = 13e^{10t}$

(g) $x'' - 4x' + 13x = 0$

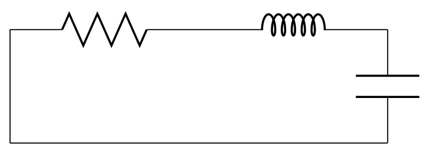
(h) $x'' + 3x' + 2x = 4t^2 - 11$

(i) $y'' - 2y' + 82y = 0$

(j) $y'' - y' = 8\sin(t)$

(k) $y''' - 6y'' + 5y' = 0$

236. *RLC circuit:* The current $i(t)$ in the circuit



satisfies the second-order differential equation

$$\frac{\mathrm{d}^2 i}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{LC}i = 0.$$

Using $R = 6\ \Omega$, $L = 2$ H, and $C = 0.04$ F, solve the IVP

$$i'' + 3i' + \frac{25}{2}i = 0, \qquad i(0) = \frac{1}{10}\text{ ampere}, \quad i'(0) = 0\ \frac{\text{ampere}}{\text{second}}.$$

237. Solve the IVP $y'' - 9y = -32\,t\,e^t, \quad y(0) = 5,\ y'(0) = \frac{1}{2}$.

238. Complete the following table:

Linear ODE	**Constant coefficients?**	**Homogeneous?**
$ty'' + \sin(t)y = 0$	no	yes
$y'' - 5y' - y = 0$		
$y'' - 5y' = y$		
$x'' + tx' - 7x = 0$		
$x'' = x + t$		
$x' = \cos(t)$		
$x' = \cos(t)x$		

239. Solve the ODE

$$y' + 17y = 0$$

for $y(t)$ using...

(a) separation of variables (this is a separable ODE).

(b) characteristic polynomials (this is a homogeneous linear ODE with constant coefficients).

240. Solve the IVP

$$x' + 3x = 8, \qquad x(0) = 9$$

in several ways:

(a) Separation of variables (this is separable).

(b) Variation of parameters (this is first-order linear).

(c) Integrating factor (this is first-order linear).

(d) The "big formula" for first-order linear ODEs.

(e) Laplace transformations.

(f) Characteristic polynomials (this is a non-homogeneous linear IVP with constant coefficients, so you will also need x_{NH} for the polynomial $g(t) = 8$).