Analysis 1

4 June 2024

Warm-up: Fill in the blank: $(\_\_\_)' = x^3$

Anti-derivatives

Last time

The words “anti-derivative” and “integral” are closely related, but for today we will only talk about anti-derivatives.

- The derivative of $9x^4$ is $36x^3$.
- The function $9x^4$ is an **anti-derivative** of $36x^3$.
- The function $9x^4 - 7$ is *also* an anti-derivative of $36x^3$.
- If we wanted to describe *all* anti-derivatives of $9x^4$ we could write

$$36x^3 + C.$$

Last time

$$\frac{1}{4}x^4 \longrightarrow x^3 \longrightarrow 3x^2 \longrightarrow 6x$$

Also x^3+7, $x^3-9.2$, etc., but we just write one simple anti-d. for this slide.

$$-\cos(x) \longrightarrow \sin(x) \longrightarrow \cos(x)$$

$$-\ln(x) \longrightarrow -x^{-1} \longrightarrow x^{-2} \longrightarrow -2x^{-3}$$

surprising, but true

Log

Historically, logarithms were created in the early 1600s as a way to multiply big numbers easily (of course, today we have calculators to do that).

Logarithms are "an inverse of exponents", but this can be misleading.

- $(\_\_) \times 5 = 40$ and $6 \times (\_\_) = 4$ are both answered using division.
- $(\_\_)^3 = 216$ is answered using a (cube) **root**.
  - $\sqrt[3]{216}$ is *by definition* the number for that blank. In this example, it's 6.
- $3^{(\_\_)} = 81$ is answered using a (base 3) **logarithm**!
  - $\log_3 81$ is *by definition* the number for that blank. In this example, it's 4.

$\sqrt[n]{x} = y$ means that $x = y^n$.

- By thinking carefully about powers, we can find properties of roots.
  - For example, $\sqrt[n]{kx} = \sqrt[n]{k} \cdot \sqrt[n]{x}$ because $(kx)^n = k^n x^n$.

$a = \log_b c$ means that $b^a = c$.

- By thinking carefully about powers, we can find properties of logs.
  - For example,

$$\log_b(cd) = \log_b(c) + \log_b(d)$$

    because $b^c b^d = b^{(c+d)}$.
  - Also, $\log_b(c^d) = d \log_b(c)$.

e

$e = 2.71828\ldots$ is a number that often appears when studying

- probability,
- compound interest,
- complex numbers,
- analysis,

and more.

One definition is $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$.

- More generally, $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$ is exactly e^x.

e and ln

$\sqrt[n]{x}$ is a root, and $\sqrt{x}$ without a superscript means $\sqrt[2]{x}$.

$\log_b(x)$ is a logarithm, but $\log(x)$ without a subscript is ambiguous (its meaning is not clear). The most common bases are 10 (in high school), 2 (in computer science), and e (in mathematics).

- $\ln(x)$ always means $\log_e(x)$.

Using the definition of $\ln$, we can see that

$$e^{\ln(x)} = x.$$

This is just like $\left(\sqrt{x}\right)^2 = x$.

Derivative formulas

Memorize these!

$f(x)$	$f'(x)$
x^p	$p\,x^{p-1}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
new! e^x	e^x
new! $\ln(x)$	$1 / x$

$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\tan(x)$	$\sec(x)^2$
a^x	$a^x \ln(a)$

Maybe these too.

Give a formula for an anti-derivative of

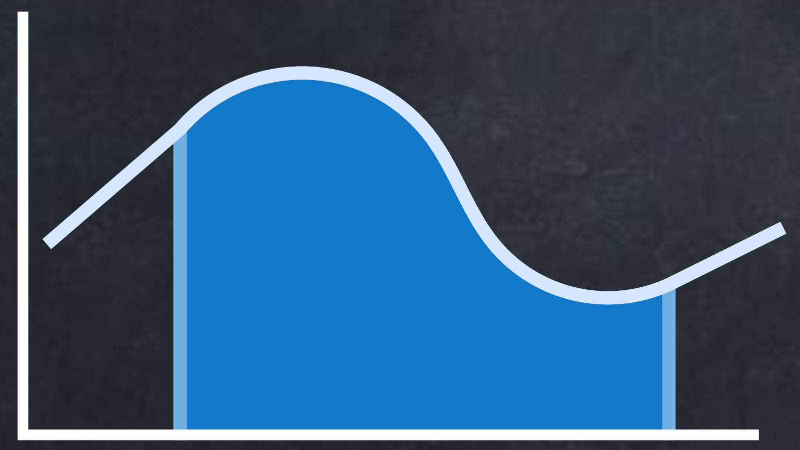
- $10x^4$ ⟶ $2x^5$
- x^{22}
- x^{-15}
- x^{-1}
- e^x
- ☆ $\ln(x)$ Answer: x ln(x) - x. But this is too hard for now.
- $\sin(x)$
- $\sin(6x)$
- ☆ $\sin(x^2)$ Literally impossible (for anyone, forever).

Area

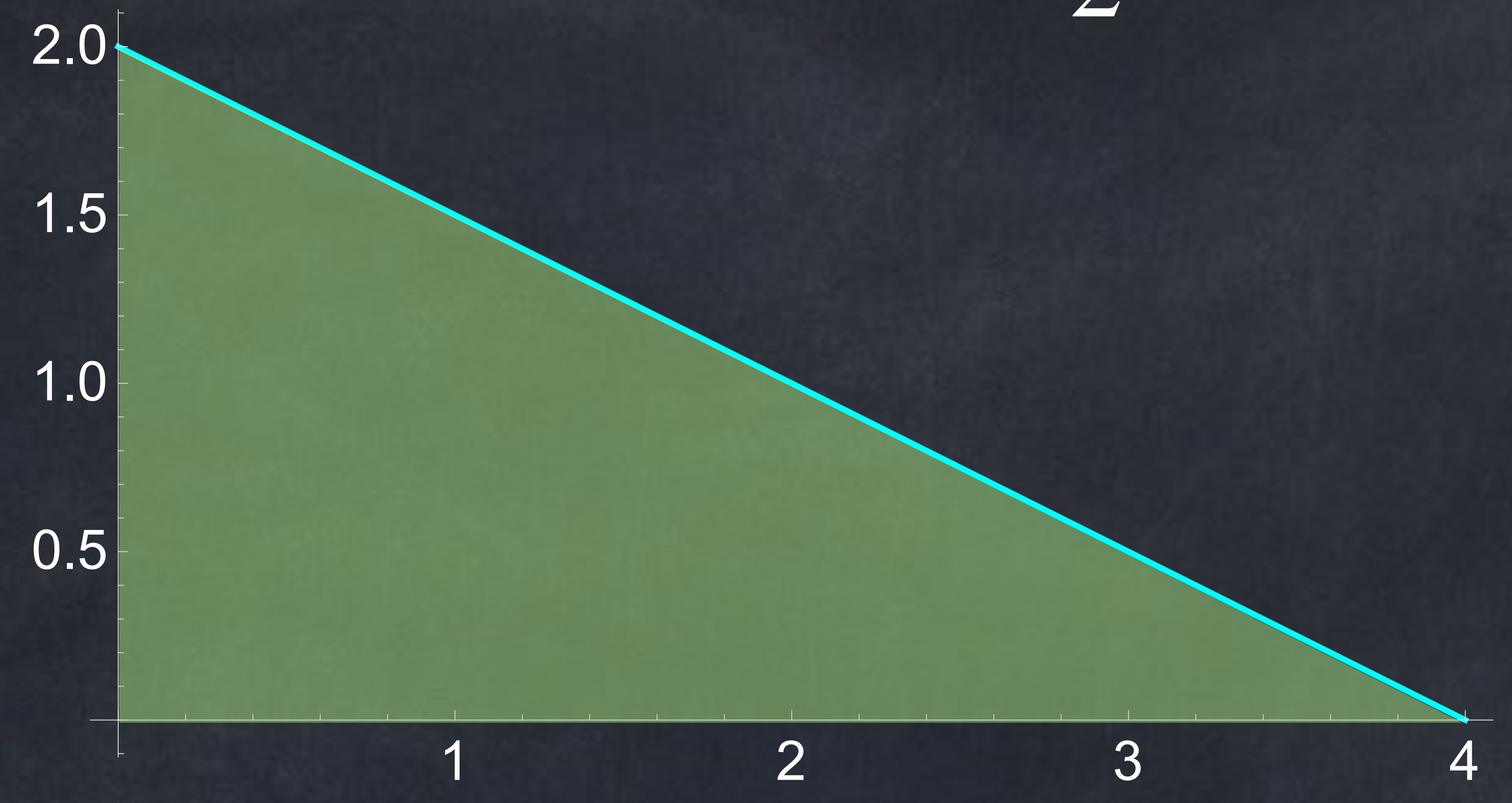
The area of a rectangle is length times width.

What about other shapes?

It is often important to calculate the area "under $y = f(x)$".
What does this mean?

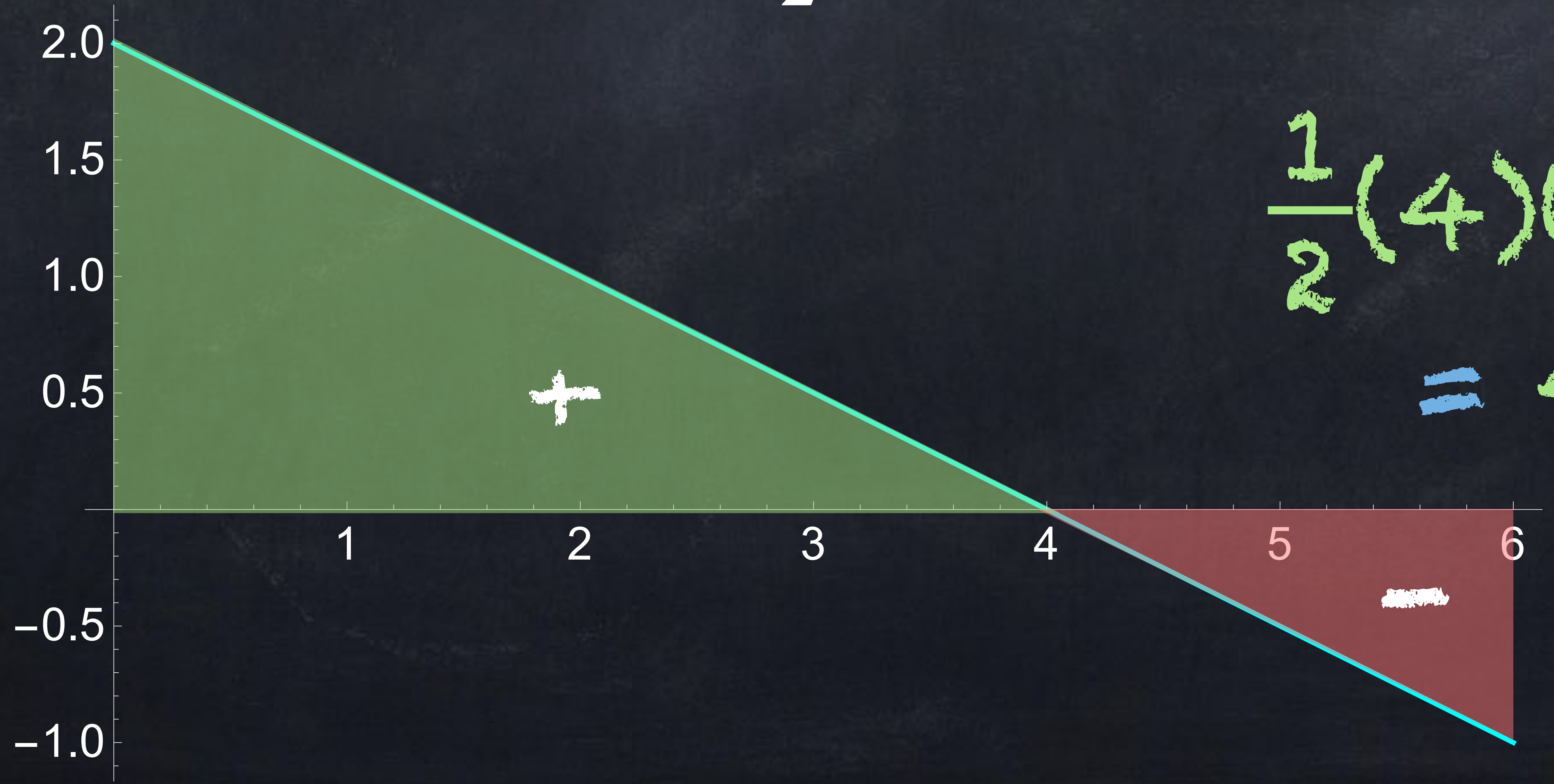
- Often this looks like  or like  $(a \leq x \leq b)$.
- However, the standard meaning of "area under $y = f(x)$" is that when $f(x) < 0$ we count this as "negative area".

Example: The "area under $y = 2 - \frac{1}{2}x$ from $x = 0$ to $x = 4$" is



$$\frac{1}{2}(\text{height})(\text{base}) = \frac{1}{2}(4)(2) = 4$$

Example: The "area under $y = 2 - \frac{1}{2}x$ from $x = 0$ to $x = 6$" is



$$\frac{1}{2}(4)(2) + \frac{1}{2}(-1)(2) = 4 - 1 = 3$$

Definite Integrals

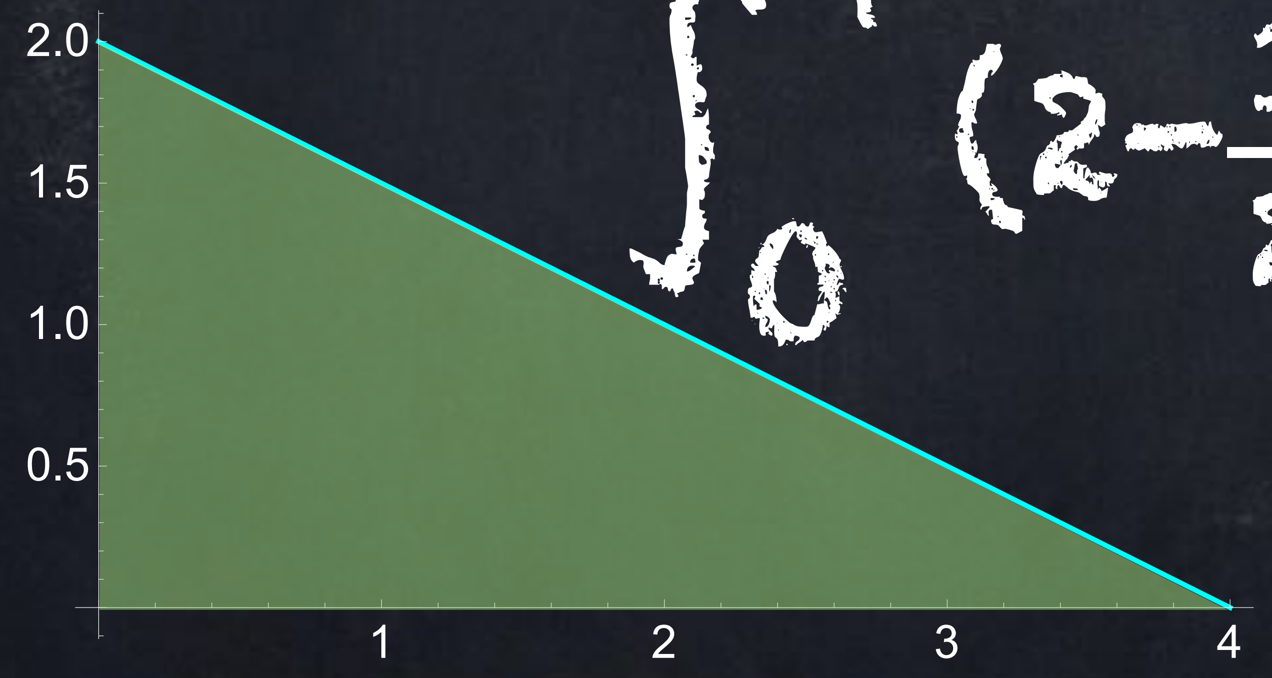
We write

$$\int_a^b f(x)\,\mathrm{d}x$$

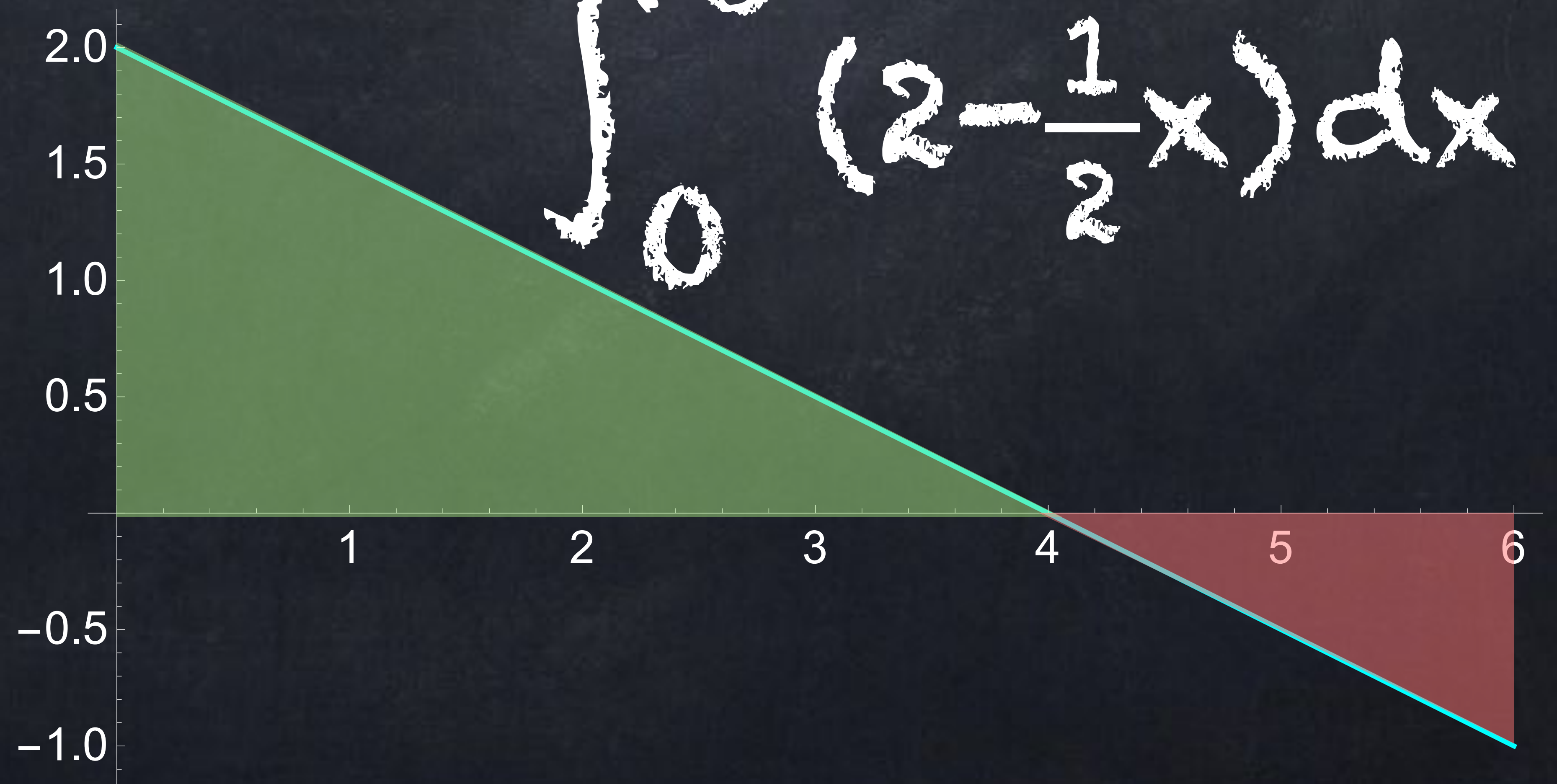
"the integral of f from a to b"

for the area under $y = f(x)$ between $x = a$ and $x = b$.

$$\int_0^4 \left(2 - \frac{1}{2}x\right) dx = 4$$



$$\int_0^6 \left(2 - \frac{1}{2}x\right) dx = 3$$



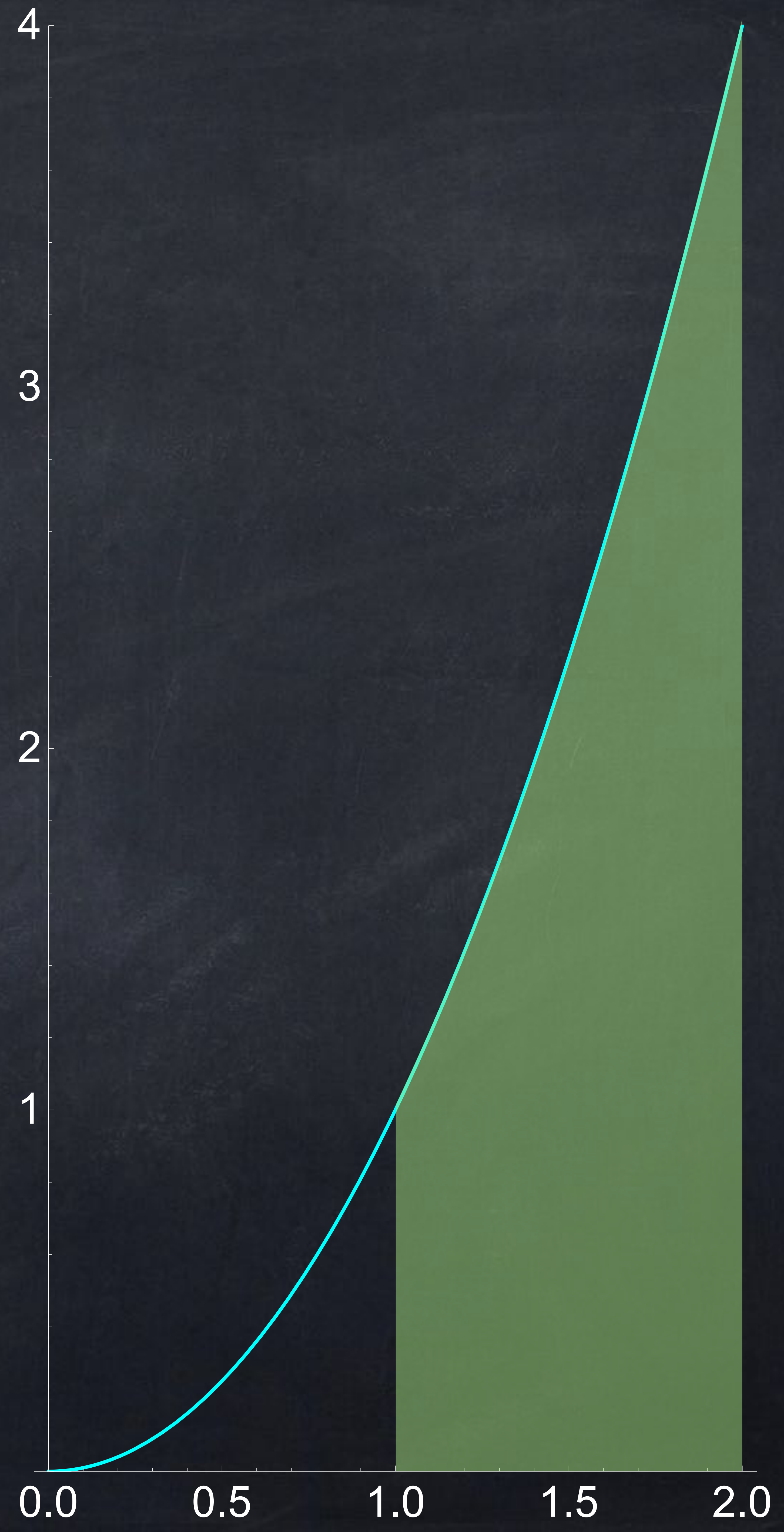
The area under $y = 3$ or $y = x$ or $y = 2 - \frac{1}{2}x$ can be calculated using rectangles and triangles.

The area under $y = x^2$ from $x = 1$ to $x = 2$ is

$$\int_1^2 x^2 \, \mathrm{d}x,$$

but what is this number?

For more complicated functions, we can approximate the area under $y = f(x)$ with rectangles.



The area $\int_1^2 x^2 \mathrm{d}x$ is *approximately* $\sum_{k=1}^{10} \frac{1}{10}\left(1+\frac{k}{10}\right)^2$ and

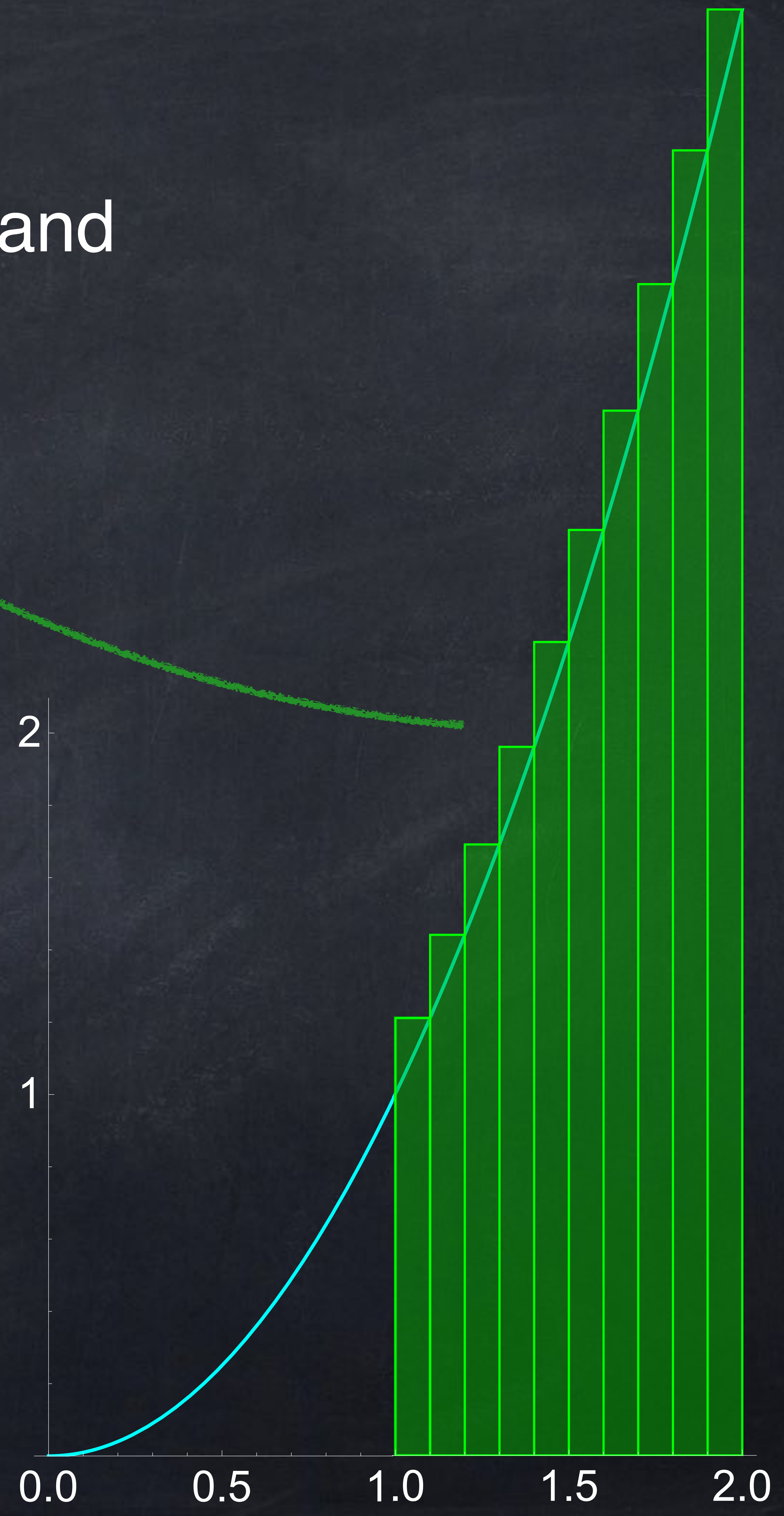
is *exactly* $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\left(1+\frac{k}{n}\right)^2$.

- With some work is possible to show that

$$\sum_{k=1}^{n} \frac{1}{n}\left(1+\frac{k}{n}\right)^2 = \frac{14n^2+9n+1}{6n^2}$$

so the area is $\frac{14}{6} = \frac{7}{3}$.

- But there is a much easier way!



Area from anti-derivatives

Instead of using a limit of a sum, there is a very nice way to compute area under a curve using an anti-derivative:

The Fundamental Theorem of Calculus

If f is continuous, then $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$,

where $F(x)$ is any function for which $F'(x) = f(x)$.

Because integrals "undo" derivatives, they appear in many places in science and engineering.

$$\text{voltage} = \int E\,dx$$

$$\text{work} = \int F\,dx$$

$$\text{impulse} = \int F\,dt$$

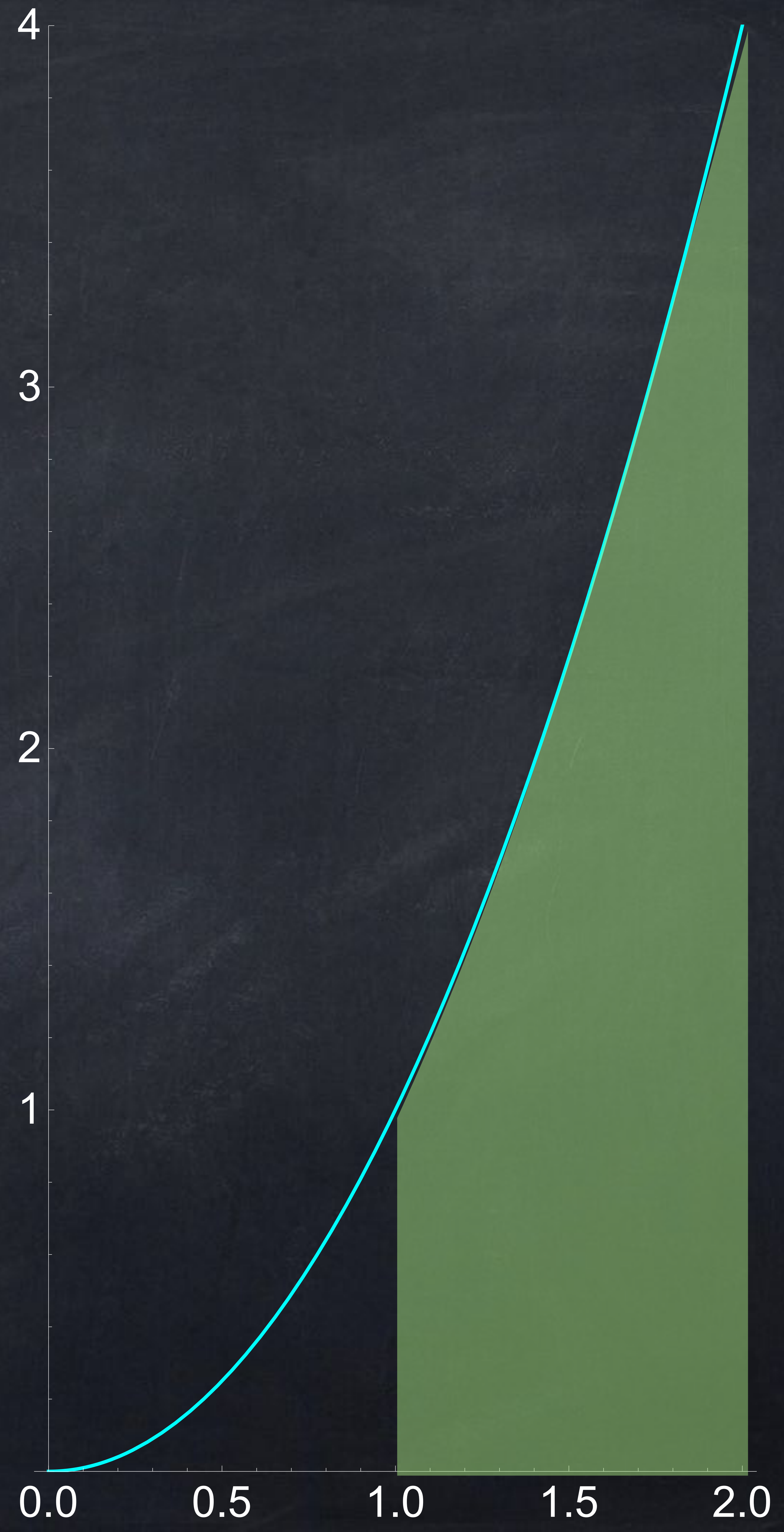
To calculate $\int_1^2 x^2 \, \mathrm{d}x$ exactly, we need a function whose derivative is x^2.

- The fact that $\frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x$ does not matter. We need the opposite idea.

$F(x) = \frac{1}{3}x^3$ satisfies $F' = x^2$

$F(2) - F(1) = \frac{1}{3}(2)^3 - \frac{1}{3}(1)^3 = \boxed{\frac{7}{3}}$

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$
with $F' = f$



Subtraction

Because we do $F(b) - F(a)$ so often, it is helpful to have a shorter way to write this. The notation

$$g(x)\Big|_{x=a}^{x=b} \quad \text{or} \quad g(x)\Big|_{a}^{b}$$

not the same as $\int_a^b g\,dx$

means $g(b) - g(a)$.

This is NOT an integral. It is just subtraction.

Example: Calculate $\cos(x)\Big|_0^{\pi}$.

$\cos(\pi) - \cos(0) = (-1) - (1) = -2$

$$\int_a^b f(x)\,\mathrm{d}x = F(x)\Big|_{x=a}^{x=b} \quad \text{with } F' = f$$

Task 1: Calculate $\int_{-5}^{4} \frac{1}{3}x^2\,\mathrm{d}x = \frac{1}{9}x^3 \Big|_{x=-5}^{x=4}$

FTC

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

with $F' = f$

$= \frac{1}{9}(4)^3 - \frac{1}{9}(-5)^3$ ← This is $F(b) - F(a)$.

$= \frac{64}{9} - \frac{-125}{9}$

$= \frac{189}{9} = \boxed{21}$

The properties below can be explained (and therefore easily remembered!) by thinking of area or thinking of anti-derivatives.

Assume f, g are functions, and a, b, c are numbers.

- $\int_a^b f \, dx + \int_b^c f \, dx = \int_a^c f \, dx$
- $\int_a^b (c \cdot f) \, dx = c \cdot \int_b^a f \, dx$
- $\int_a^b f \, dx + \int_a^b g \, dx = \int_a^b (f + g) \, dx$
- $\int_a^b f \, dx = -\int_b^a f \, dx$

Task 2: Calculate $\int_0^5 |x-2|\,\mathrm{d}x$.

$$\int_0^2 (2-x)\,dx + \int_2^5 (x-2)\,dx = \cdots$$

Final answer: 13/2

Task 3: Calculate $\int_0^{6\pi} f(x)\,\mathrm{d}x$ for the function $f(x) = \begin{cases} e^{x-\pi} & \text{if } x \leq \pi \\ 2\cos(x) & \text{if } x > \pi. \end{cases}$

Final answer: $1 - e^{-\pi}$

Definite vs. Indefinite ∫

The integrals we have done so far are examples of "definite integrals".

- Definite: $\int_1^2 x^2 \, \mathrm{d}x = \frac{7}{3}$

In order to calculate this, we needed to use $\frac{1}{3}x^3$.

An **indefinite integral** is just a way of writing all the anti-derivatives of a function. We use the $\int$ symbol but do not put any numbers at the top or bottom.

- Indefinite: $\int x^2 \, \mathrm{d}x = \frac{1}{3}x^3 + C$

Notation (how to write math)

	Newton	Leibniz	Euler / Lagrange
Derivative of f	$\dot{f}$	$\frac{\mathrm{d}f}{\mathrm{d}x}$	f' or $f^{(1)}$
Anti-derivative of f	$\overset{\mid}{f}$ or $\boxed{f}$	$\int f \,\mathrm{d}x$	$f^{(-1)}$

All the ways of writing derivatives are still common today.

Only $\int f \,\mathrm{d}x$ is common for anti-derivatives.

Integral techniques

The four most common types of integrals are:

- "basic" (just think about derivatives) ← this is the kind we have done today,
- algebra-first,
- substitution,
- parts.

Often the most difficult part of an integral is deciding which technique to use.
The best way to improve is practice!