Analysis 1

11 June 2024

Warm-up: (a) If $w = 3t^8$, what is $\frac{\mathrm{d}w}{\mathrm{d}t}$?

(b) If $m = x^5 e^{3x}$, what is $\frac{\mathrm{d}m}{\mathrm{d}x}$?

Definite vs. Indefinite ∫

Last time

A "definite integral" can be thought of as calculating an area.

- Definite: $\int_1^2 x^2\,\mathrm{d}x = \frac{7}{3}$

According the to FUNDAMENTAL THEOREM OF CALCULUS, we can use the anti-derivative $\frac{1}{3}x^3$ to calculate this: $\frac{1}{3}(2)^3 - \frac{1}{3}(1)^3$.

An **indefinite integral** is just a way of writing all the anti-derivatives of a function. We use the $\int$ symbol but do not put anything above or below.

- Indefinite: $\int x^2\,\mathrm{d}x = \frac{1}{3}x^3 + C$

Last time

The Fundamental Theorem of Calculus

If f is continuous then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a),$$

where $F(x)$ is any function for which $F'(x) = f(x)$.

The subtraction on the right can also be written $F(x)\Big|_{x=a}^{x=b}$ or $\Big[F(x)\Big]_{x=a}^{x=b}$.

$$\int \sin(x)\,\mathrm{d}x = -\cos(x) + C$$

$$\int \sin(t)\,\mathrm{d}t = -\cos(t) + C$$

$$\int t^8\,\mathrm{d}t = \frac{1}{9}t^9 + C$$

Power Rule

Derivatives: $\dfrac{\mathrm{d}}{\mathrm{d}x} x^n = n\, x^{n-1}$

Integrals: $\displaystyle\int x^n \,\mathrm{d}x = \frac{1}{n+1} x^{n+1} + C$ if $n \neq -1$

$\displaystyle\int x^{-1} \,\mathrm{d}x = \ln(x) + C$

Last time

The properties below can be explained—and therefore easily remembered!—by thinking of (signed) area or thinking of anti-derivatives.

Assume f, g are functions, and a, b, c are constants.

- $\int_a^b f\,\mathrm{d}x + \int_b^c f\,\mathrm{d}x = \int_a^c f\,\mathrm{d}x$
- $\int_a^b (c \cdot f)\,\mathrm{d}x = c \cdot \int_a^b f\,\mathrm{d}x$
- $\int_a^b f\,\mathrm{d}x + \int_a^b g\,\mathrm{d}x = \int_a^b (f+g)\,\mathrm{d}x$
- $\int_a^b f\,\mathrm{d}x = -\int_b^a f\,\mathrm{d}x$

For indefinite integrals:

- $\int (c \cdot f)\,\mathrm{d}x = c \cdot \int f\,\mathrm{d}x$

- $\int f\,\mathrm{d}x + \int g\,\mathrm{d}x = \int (f+g)\,\mathrm{d}x$

Find $\int (\cos(x) + 2x^6)\,dx = \left(\int \cos(x)\,dx\right) + 2\left(\int x^6\,dx\right)$

$= \left(\sin(x) + C_1\right) + 2\left(\frac{1}{7}x^7 + C_2\right)$

$= \boxed{\sin(x) + \frac{2}{7}x^7 + C_3}$

$C_3 = C_1 + 2C_2$

Find $\int (\cos(x) + 2x^6)\, dx = \left(\int \cos(x)\, dx\right) + 2\left(\int x^6\, dx\right)$

$$= \left(\sin(x) + C\right) + 2\left(\frac{1}{7}x^7 + C\right)$$

$$\boxed{= \sin(x) + \frac{2}{7}x^7 + C}$$

It's common to use "$+C$" for any constant, even in related formulas. The C in the last line is the C from $\sin(x) + C$ plus 2 times the C from $\frac{1}{7}x^7 + C$. Instead of writing C_1, then C_2, and then $C_1 + 2C_2$, we can just write C for all of this.

Integral techniques

The four most common types of integrals are:

- "basic" (just think about derivatives, backwards)
- algebra-first,
- substitution,
- parts.

Examples: $\displaystyle\int \frac{x^2 + 7x}{3x}\,\mathrm{d}x = \,?$ Re-write this as $\frac{1}{3}x + \frac{7}{3}$ first!

$\displaystyle\int \frac{3x}{x^2 + 7}\,\mathrm{d}x = \,?$ This needs "u-sub" (later).

$\int (f+g)\,\mathrm{d}x$ and $\int (f-g)\,\mathrm{d}x$ are easy: just add/subtract the integrals of the two functions.

$\int (fg)\,\mathrm{d}x$ and $\int (\frac{f}{g})\,\mathrm{d}x$ are very hard in general.

What can we say about $\int (fg')\,\mathrm{d}x$?

Integration by parts

There are several (equally valid) ways to write the "parts" formula.

Product rule but with integrals:

$$\int f\,g'\,\mathrm{d}x + \int f'g\,\mathrm{d}x = fg$$

$$\int f\,g'\,\mathrm{d}x = fg - \int f'g\,\mathrm{d}x$$

$$\int u\,\mathrm{d}v = u\,v - \int v\,\mathrm{d}u$$

Integration by parts

There are several (equally valid) ways to write the "parts" formula.

$$\int f g' \mathrm{d}x + \int f' g \mathrm{d}x = fg$$

$$\int f g' \mathrm{d}x = fg - \int f' g \mathrm{d}x$$

$$\int u \, \mathrm{d}v = u\,v - \int v \, \mathrm{d}u$$

Example 1: $\int 2x \cos(3x)\,dx$

Parts:

$$\int f g' \, dx = f g - \int f' g \, dx$$

Answer: $\frac{2}{3}x\sin(3x) - \frac{2}{9}\cos(3x) + C$

Task 2: $\int 4e^{2x}x^2\,dx = ? + C$

Parts:

$$\int f g'\,dx = fg - \int f' g\,dx$$

$$\int (4x^2)(e^{2x})\,dx = (4x^2)\left(\frac{1}{2}e^{2x}\right) - \int (8)\left(\frac{1}{2}e^{2x}\right)dx$$

$$= 2x^2e^{2x} - \int 4xe^{2x}\,dx$$

This requires integration by parts also!

Final answer: $(2x^2-2x+1)e^{2x} + C$

Task 1: $\int \underbrace{2x}_{f} \underbrace{\cos(3x)}_{g'} \, dx$

works the first time

Task 2: $\int \underbrace{4x^2}_{f} \underbrace{e^{2x}}_{g'} dx$

requires ∫ by parts twice

Next time: $\int e^x \cos(x) \, dx.$

(parts, but stranger)