Analysis 1

14 June 2024

Warm-up: If $\frac{a}{b} = 2x$, then

$a = ?$ $\qquad b = ?$

$a = 2xb \qquad b = \frac{2x}{a}$

Calculate $\int \underbrace{2x}_{f} \underbrace{\cos(3x)}_{g'}\,dx.$

Parts:

$$\int f g'\,dx = f g - \int f' g\,dx$$

Last time

$$\int (2x)\cos(3x)\,dx = (2x)\left(\frac{1}{3}\sin(3x)\right) - \int (2)\left(\frac{1}{3}\sin(3x)\right)dx$$

$$= \frac{2}{3}x\sin(3x) - \int \frac{2}{3}\sin(3x)\,dx$$

$$= \boxed{\frac{2}{3}x\sin(3x) + \frac{2}{9}\cos(3x) + C}$$

Task 2: $\int 4 e^{2x} x^2 \, dx = ? + C$

Parts:

$$\int f g' \, dx = f g - \int f' g \, dx$$

Not Last time

$$\int (4x^2)(e^{2x}) \, dx = (4x^2)\left(\frac{1}{2}e^{2x}\right) - \int (8)\left(\frac{1}{2}e^{2x}\right) dx$$

$$= 2x^2 e^{2x} - \int 4x e^{2x} \, dx$$

This requires integration by parts also!

Task 2: $\int 4\, e^{2x}\, x^2 \, dx = ? + C$

Parts:

$$\int f g' dx = f g - \int f' g dx$$

Not Last time

$$\int (4x^2)(e^{2x})dx = (4x^2)(\tfrac{1}{2}e^{2x}) - \int (8)(\tfrac{1}{2}e^{2x})dx$$

$$= 2x^2e^{2x} - \int 4xe^{2x}dx$$

Parts again!

$$= 2x^2e^{2x} - \left(4x\cdot\tfrac{1}{2}e^{2x} - \int \tfrac{1}{2}e^{2x}\cdot 4dx + C\right)$$

$$= 2x^2e^{2x} - \left(2xe^{2x} - e^{2x} + C\right)$$

$$= \boxed{(2x^2-2x+1)e^{2x} + C}$$

Task 3: Calculate $I = \int_0^4 3xe^x \mathrm{d}x$.

Indefinite: $\int 3xe^x dx = 3xe^x - \int 3e^x dx$ using parts

$= 3xe^x - 3e^x + C$

so the answer is $(12e^4 - 3e^4) - (0 - 3)$

$= (9e^4) - (-3)$

$= \boxed{9e^4 + 3}$

You can also use $3xe^x \Big|_0^4 - \int_0^4 3e^x dx = 9e^4 + 3$, but it's no better.

Task 4: Find $I = \int e^x \cos(x)\, \mathrm{d}x$.

$I = e^x \sin(x) - \int e^x \sin(x) dx$

Do parts again for $\int e^x \sin(x) dx$.

$$\int e^x \sin(x) dx = -e^x \cos(x) - \int -e^x \cos(x) dx$$
$$= -e^x \cos(x) + I$$

Don't do parts again (look). Instead, write equation with I.

$$I = e^x \sin(x) + e^x \cos(x) - I$$

$$2I = e^x \sin(x) + e^x \cos(x)$$

$$I = \boxed{\frac{1}{2} e^x \sin(x) + \frac{1}{2} e^x \cos(x) + C}$$

Task 1: $\int 2x\ \cos(3x)\,\mathrm{d}x$

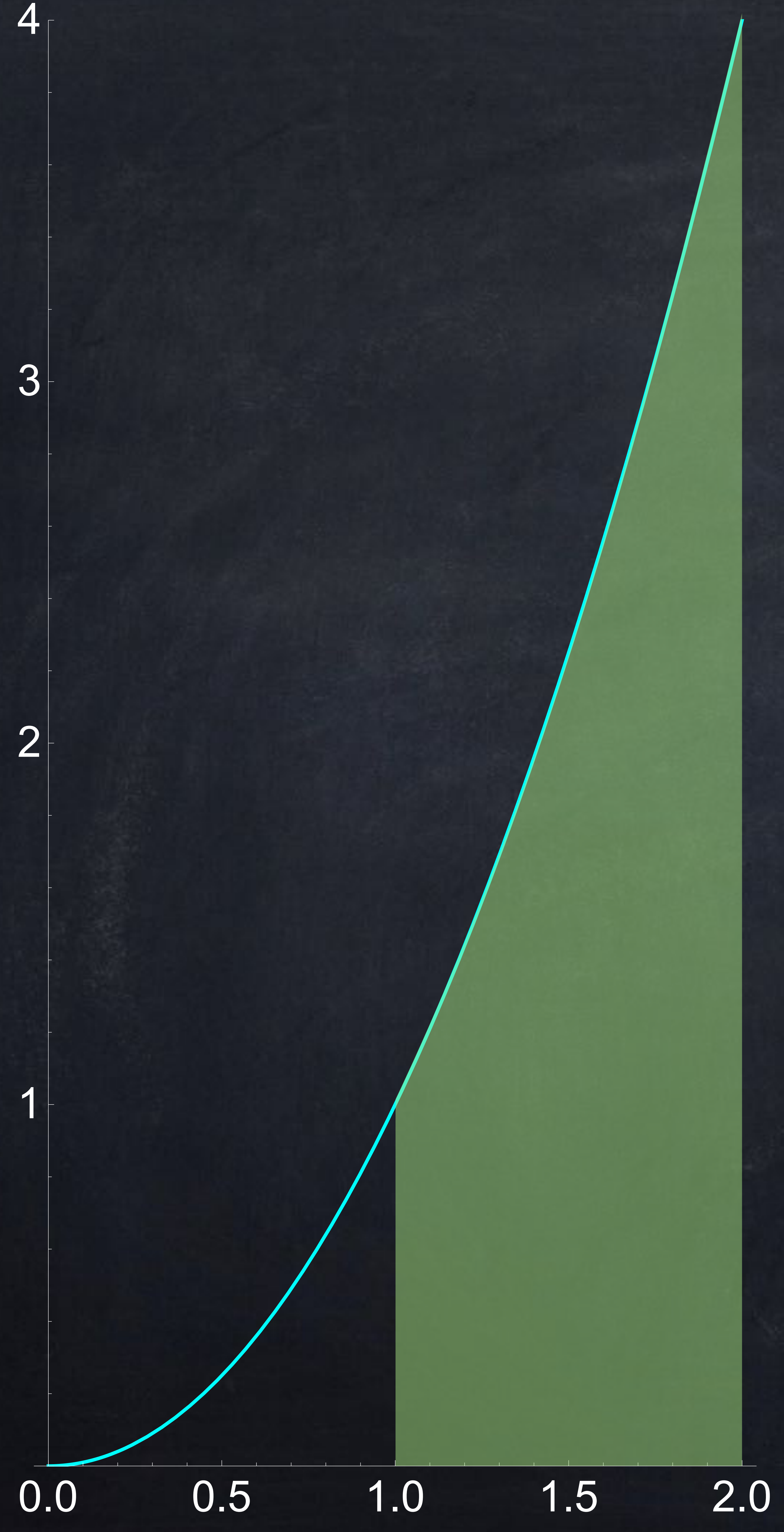
works the first time

Task 2: $\int 4\,x^2\,e^{2x}\mathrm{d}x$

requires $\int$ by parts twice

Task 4: $\int e^x \cos(x)\,\mathrm{d}x.$

parts twice, then solve eqn. for $\int$



The area under $y = x^2$ from $x = 1$ to $x = 2$ is

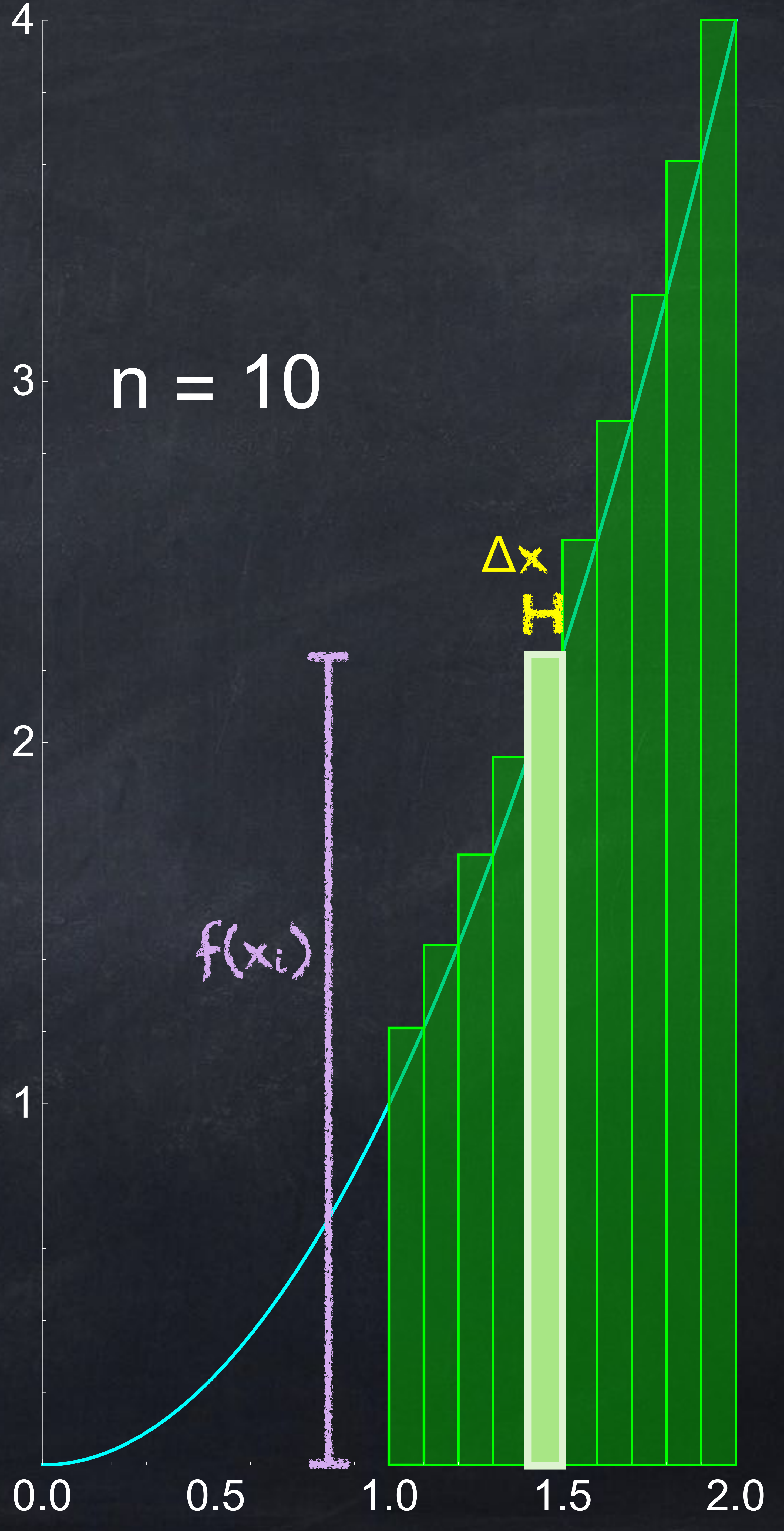
$$\int_1^2 x^2 \, dx.$$

This can be *approximated* using lots of rectangles. The exact value is

$$\lim_{n \to \infty} \sum_{k=1}^{n} \left(1 + \frac{k}{n}\right)^2 \frac{1}{n}$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} f(x_i) \Delta x$$

but can be calculated more easily using anti-derivatives.



$$\int \sin(x)\,dx = -\cos(x) + C$$

$$\int ax\,dx = \frac{a}{2}x^2 + C$$

$$\int \sqrt{x}\,dx = \int x^{1/2}dx$$

$$= \frac{2}{3}x^{3/2} + C$$

$$\int u^8\,du = \frac{1}{9}u^9 + C$$

$$\int v^8 dv = \frac{1}{9}v^9 + C$$

$$\int w^8 dw = \frac{1}{9}w^9 + C$$

$$\int x^8\,dx = \frac{1}{9}x^9 + C$$

$$\int y^8 dy = \frac{1}{9}y^9 + C$$

Find $\int (x^2 + 5)^8 (2x)\, dx$.

Hint: Use a new variable $u = x^2 + 5$.

$u = x^2 + 5$

$\frac{du}{dx} = 2x$

$du = 2x\, dx$

It may seem like cheating to pretend that du/dx is a fraction, but it's actually very helpful to say du = 2x dx because we can use this to rewrite the original integral as an integral with u.

Answer: $\frac{1}{9}(x^2+5)^9 + C$

u-substitution

When we see a function and its derivative in a certain configuration, we can re-write an integral using “substitution”.

- As a general formula, we have

$$\int f\big(g(x)\big)\,g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u, \qquad \text{where } u = g(x).$$

  but examples may be easier to understand than this formula.

- We often use u as the new variable of integration, so this method is also called “u-substitution” or just “u-sub”.

Task 1: Find $\int 6x^2 \cos(x^3 + 9)\,\mathrm{d}x$.

Using $u = x^3+9$, ...

Answer: $2\sin(x^3+9) + C$

In general, we need $f(u)$ multiplied by u' or by ku' with k constant.

$\cos(x^3+9)$ $3x^2$ $2(3x^2)$

Task 2: Find $\int \frac{1}{x\ln(x)}\mathrm{d}x$.

Using $u = \ln(x)$, ...

Answer: $\ln(\ln(x)) + C$

In general, we need $f(u)$ multiplied by u' or by ku' with k constant.

1/(ln x)   1/x