Analysis 1

18 June 2024

Warm-up: If $u = 2x$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = 2$ and $\mathrm{d}u = 2\mathrm{d}x$.

If $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos(3x)$ then $v = \frac{1}{3}\sin(3x)$.

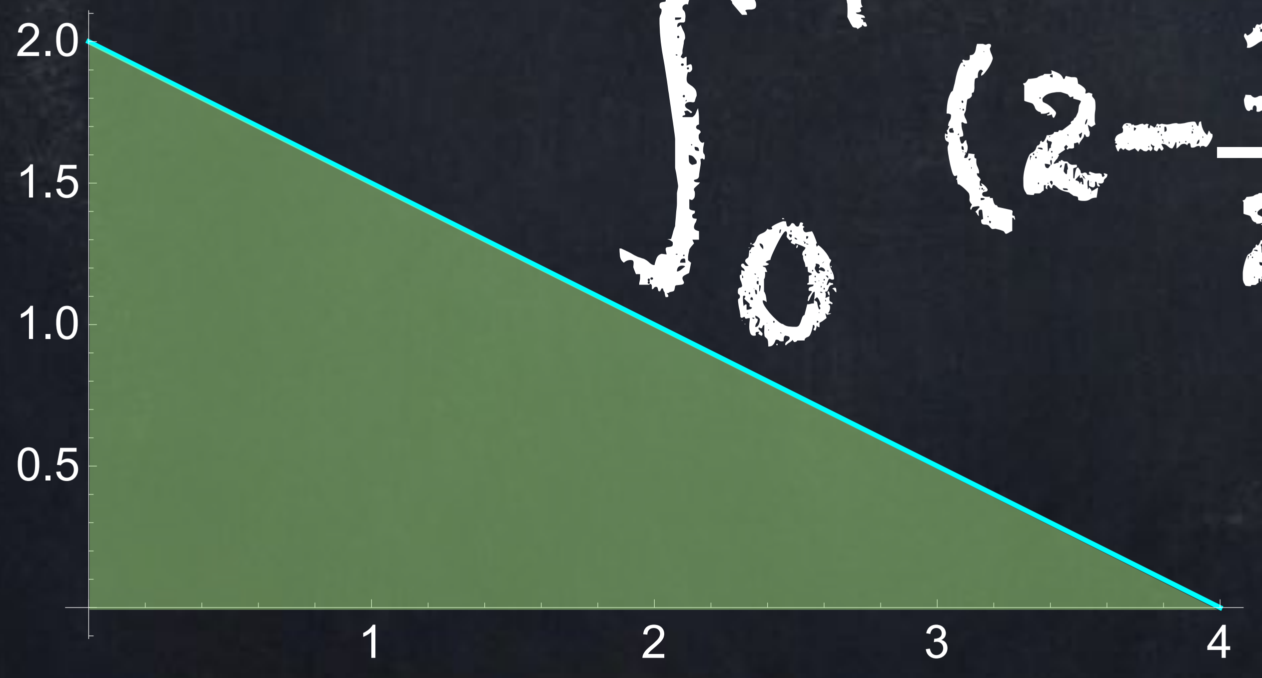
Previously

A definite integral, like

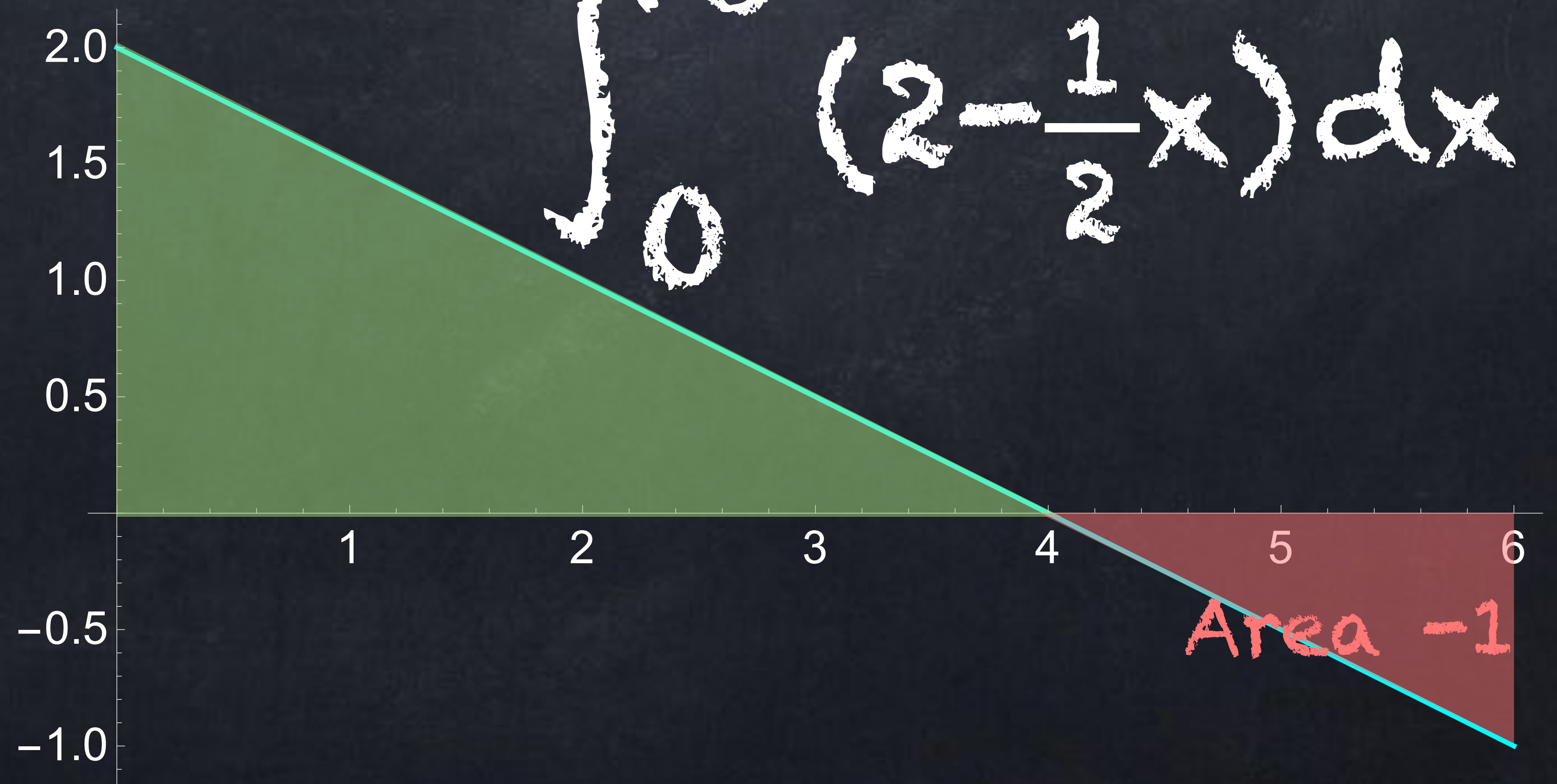
$$\int_a^b f(x)\,\mathrm{d}x$$

"the integral of f from a to b"

can be thought of as the "area under $y = f(x)$" between $x = a$ and $x = b$, but this is technically *signed area*: when $f(x) < 0$ we count negative area.



$$\int_0^4 \left(2-\frac{1}{2}x\right)dx = 4$$



$$\int_0^6 \left(2-\frac{1}{2}x\right)dx = 3$$

Previously

The Fundamental Theorem of Calculus

If f is continuous then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a),$$

where $F(x)$ is any anti-derivative of $f(x)$.

An indefinite integral is just a way of asking for all the anti-derivatives:

- Example: $\int (2-\frac{1}{2}x)\,\mathrm{d}x \quad = \quad 2x - \frac{1}{4}x^2 + C$

The numbers at the top and bottom of the integral symbol in a definite integral are called **bounds**. Indefinite integrals don’t have bounds.

New vocab

Previously

parts

To use this we need $\int fg'\mathrm{d}x.$ $= fg - \int f'g\,dx$

u-sub

To use this we need $\int f(u)u'\mathrm{d}x.$ $= \int f(u)\,du$

or $\int f(u)3u'\mathrm{d}x$ or similar.

Task 1: Find $\displaystyle\int \frac{24x}{\sqrt{4x^2-1}}\,\mathrm{d}x.$

Using $u = 4x^2 - 1$, ...

Answer: $6\sqrt{4x^2-1} + C$

Task 2: Calculate $\int_{1/2}^{(\sqrt{5})/2} \frac{24x}{\sqrt{4x^2-1}}\,\mathrm{d}x$..

FTC

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

where $F'(x) = f(x)$.

Option 1: Using the final answer to the previous task, this is $6\sqrt{4x^2-1}\Big|_{x=1/2}^{x=(\sqrt{5})/2} = \ldots$

Option 2: Change x-values into u-values for the bounds: $\int_0^4 3u^{-1/2}du = 6u^{1/2}\Big|_{u=0}^{u=4} = 12.$

Task 3: Find $\int_0^1 2(x^2+3)^2\,\mathrm{d}x = \int_0^1 (2x^4 + 12x^2 + 18)dx$

NOT U-SUB!

Answer: 22.4

For many students the most difficult thing about integrals is recognizing *when* to use parts, u-sub, or other methods. The best way to improve is practice!

Task 4: Evaluate $\int_1^7 \frac{x}{x^2+1} \mathrm{d}x$.

Using $u = x^2 + 1$, ...

Answer: $\frac{1}{2}\ln(50) - \frac{1}{2}\ln(2)$

or $\frac{1}{2}\ln(25)$

or $\ln(5)$

You should know

$e^0 = 1 \qquad \ln(1) = 0$

$e^1 = e \qquad \ln(e) = 1$

$e^{\ln(x)} = x \qquad \ln(e^x) = x \qquad (\sqrt{x})^2 = x \qquad \sqrt{x^2} = x$ if $x \geq 0$

Integration by parts

$$f g' + f'g = (fg)'$$

$$f g' = (fg)' - f'g$$

$$\int f g'\mathrm{d}x = fg - \int f'g\,\mathrm{d}x$$

Last week's version.
To use this, we choose f and g so that our original integral is $\int(fg')\mathrm{d}x$.

Alternative.
To use this, we choose u and dv so that our original integral is $\int u\mathrm{d}v$.

$$\int u\,\mathrm{d}v = u\,v - \int v\,\mathrm{d}u$$

ultraviolet voodoo
might help you remember this formula

Example from last week: $\int \underbrace{2x}_{u}\,\underbrace{\cos(3x)\,\mathrm{d}x}_{dv}$

Parts:

$$\int u\,\mathrm{d}v = u\,v - \int v\,\mathrm{d}u$$

$$\int (2x)\cos(3x)dx = (2x)\left(\frac{1}{3}\sin(3x)\right) - \int \left(\frac{1}{3}\sin(3x)\right)(2)dx$$

$$= \frac{2}{3}x\sin(3x) - \int \frac{2}{3}\sin(3x)dx$$

$$= \boxed{\frac{2}{3}x\sin(3x) + \frac{2}{9}\cos(3x) + C}$$

Task 5: $\int e^{8x} x \mathrm{d}x$

$u = x$ $f = x$

same as

$dv = e^{8x}dx$ $g' = e^{8x}$

Answer: $\frac{1}{8}xe^{8x} - \frac{1}{64}e^{8x} + C$ or $\frac{1}{64}(8x - 1)e^{8x} + C$

Task 5: $\int \ln(x)\,\mathrm{d}x$

$u = \ln(x)$

$f = \ln(x)$

same as

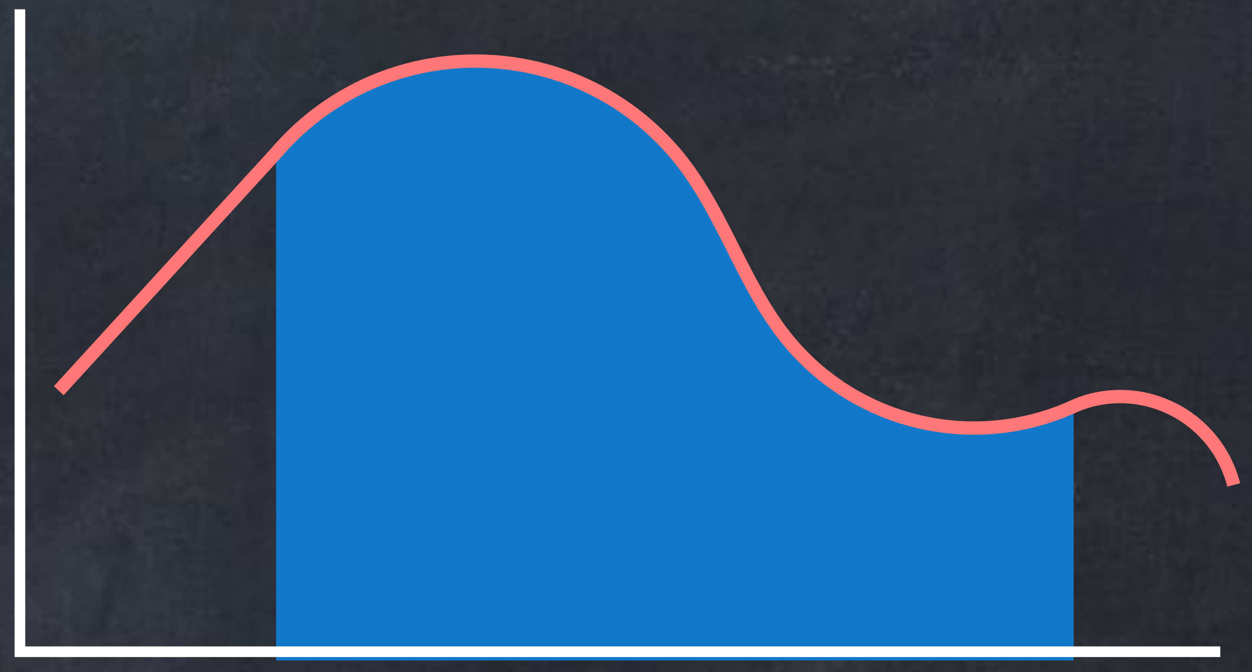
$dv = dx$

$g' = 1$

Answer: $x\ln(x) - x + C$

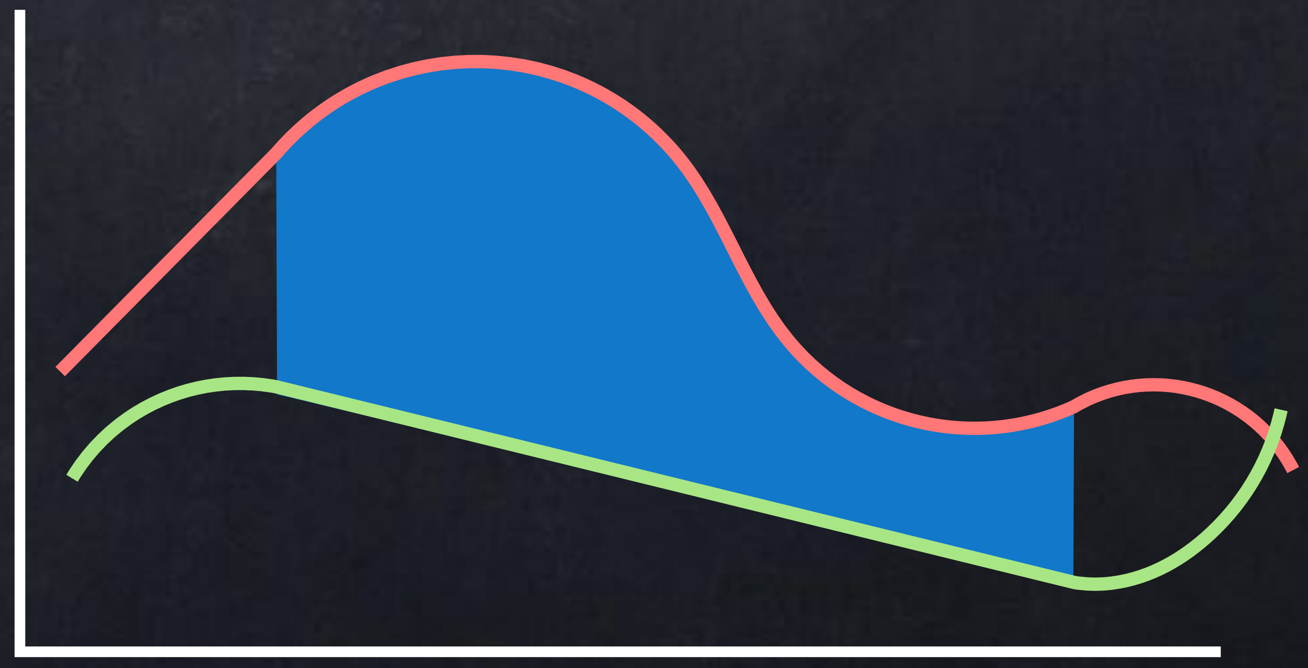
Area between curves

Previously we defined $\int_a^b f(x)\mathrm{d}x$ as the area between $y = f(x)$ and the x-axis with $a \leq x \leq b$.

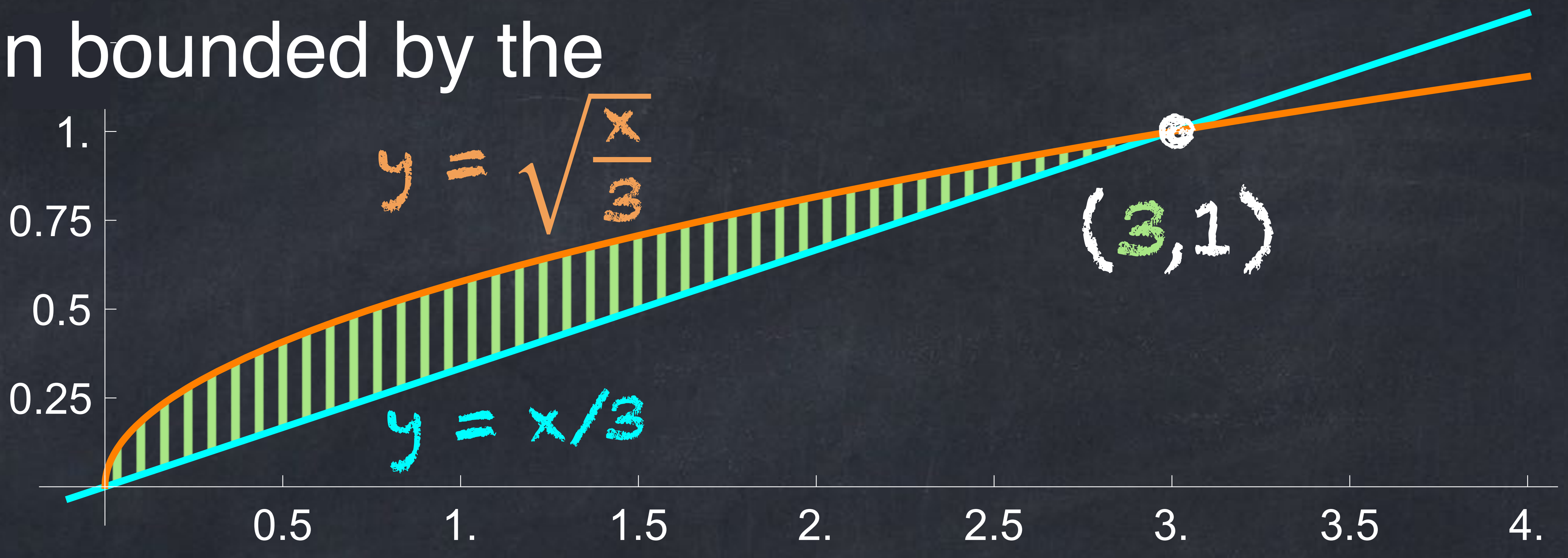


If $f(x) \leq g(x)$ for all $a \leq x \leq b$ then the area between the curves $y = f(x)$ and $y = g(x)$ with $a \leq x \leq b$ is

$$\int_a^b \big(g(x) - f(x)\big)\mathrm{d}x.$$



Example: Find the area of the region bounded by the curves $\underbrace{y = \frac{x}{3}}_{\text{bottom}}$ and $\underbrace{x = 3y^2}_{\text{top}}$.



$$\text{Area} = \int_{\text{left}}^{\text{right}} (\text{top} - \text{bottom})\,dx = \int_0^3 \left(\frac{1}{\sqrt{3}}x^{1/2} - \frac{1}{3}x\right)dx$$

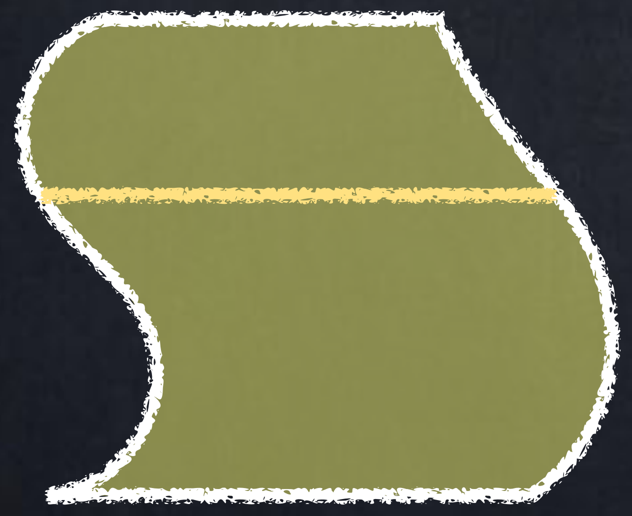
$$= \dots = \frac{1}{2}$$

Area is integral of height or width

The area of a shape with $a \le x \le b$ and with curves on the top and bottom is

$$\int_a^b \Big(\underline{\mathrm{Top}(x) - \mathrm{Bottom}(x)}\Big)\,\mathrm{d}x.$$

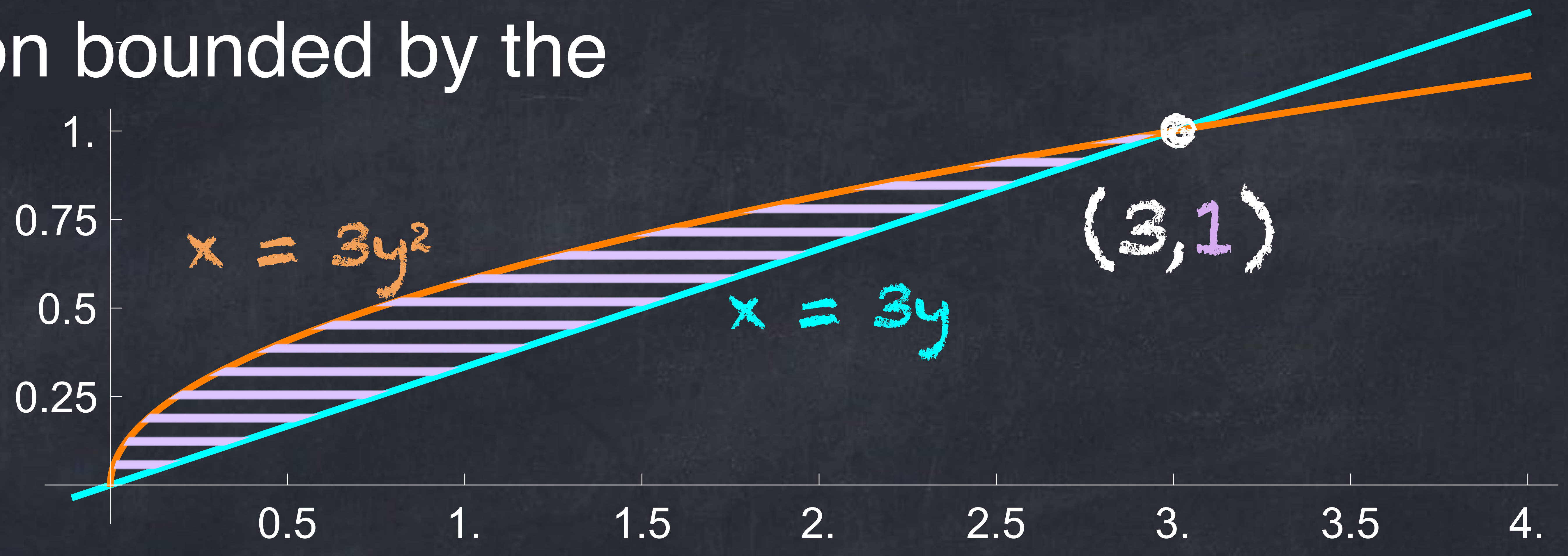
The area of a shape with $c \le y \le d$ and with curves on the left and right is



$$\int_c^d \Big(\underline{\mathrm{Right}(y) - \mathrm{Left}(y)}\Big)\,\mathrm{d}y.$$

For some shapes, both methods are possible!

Example: Find the area of the region bounded by the curves $\underbrace{y = \frac{x}{3}}_{\text{right}}$ and $\underbrace{x = 3y^2}_{\text{left}}$.



We can use EITHER

$$\int_L^R (\text{Height})\,dx = \int_L^R (\text{Top} - \text{Bottom})\,dx = \int_0^3 \left(\sqrt{\frac{x}{3}} - \frac{x}{3}\right) dx = \frac{1}{2}$$

or

$$\int_B^T (\text{Width})\,dy = \int_B^T (\text{Right} - \text{Left})\,dy = \int_0^1 (3y - 3y^2)\,dy = \frac{1}{2}$$